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Supersymmetries in Free Fermionic Strings

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Abstract

Consistent heterotic free fermionic string models are classified in terms of their number of spacetime supersymmetries, N . For each of the six distinct choices of gravitino sector, we determine what number of supersymmetries can survive additional GSO projections. We prove by exhaustive search that only three of the six can yield $N = 1$, in addition to the $N = 4, 2$, or 0 that five of the six can yield. One choice of gravitino sector can only produce $N = 4$ or 0 . Relatedly, we find that only \mathbb{Z}_2 , \mathbb{Z}_4 , and \mathbb{Z}_8 twists of the internal fermions with worldsheet supersymmetry are consistent with $N = 1$ in free fermionic models. Any other twists obviate $N = 1$.

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1. Review of Free Fermionic Strings

Over the last decade, string model building has grown into a field with sometimes parallel, sometimes intersecting avenues. Essentially these avenues can be viewed as the various approaches to “compactification” from ten-dimensions to four. Some of which lead to the geometrical interpretation of actual compactified dimensions, while others do not. The primary avenues correspond to bosonic lattices,^[1,2] free fermions,^[3,4] \mathbb{Z}_n -orbifolds,^[5–7] Calabi-Yau manifolds,^[8,9] and $N = 2$ minimal models.^[10] Some of these avenues intersect with others only at a few locations; some overlap with others significantly. Some turn out to be sections of a larger avenue; some are eventually found to be identical to another, covering the same path, as a single road given two different names. Along varying avenues of string models and varying the locations on a specific avenue, the number, N , of spacetime supersymmetries in the early stringy universe can be as large as 8 and as small as 0, depending on location. Recent LEP results, however, point strongly towards the sections of these various avenues that yield exactly $N = 1$ spacetime supersymmetry (ST-SUSY).^[11] In this paper we traverse one specific avenue, that of free fermions (and in particular the sub-avenue of heterotic free fermions), and search for the sections along this path that do, indeed, produce $N = 1$ spacetime supersymmetry.

Free fermionic model building was developed simultaneously by Antoniadis, Bachas, Kounas, and Windey in [3] and by Kawai, Lewellen, and Tye in [4]. In light-cone gauge, a heterotic free fermionic string model contains 64 real worldsheet (WS) fermions ψ^n ($1 \leq n \leq 20$ for left-moving (LM) WS fermions, $21 \leq n \leq 64$ for right-moving (RM) WS fermions), in addition to the LM/RM WS scalars ($X_{\mu=1,2}, \bar{X}_{\bar{\mu}=1,2}$) embedding transverse coordinates of four-dimensional spacetime. ψ^1 and ψ^2 are the WS superpartners of the two LM transverse scalars; the remaining 62 fermions, $\{\psi^{n=3 \text{ to } 64}\}$ are internal degrees of freedom. Some or all of these internal fermions may be paired to form complex fermions $\psi^{n,m} \equiv \psi^n + i\psi^m$. If n and m both denote to either left-movers or right-movers, then $\psi^{n,m}$ is a Weyl fermion. If n denotes a left-mover and m a right-mover, then $\psi^{n,m}$ is a Majorana fermion. A specific model is defined by (1) sets of 64-component boundary vectors (with components for complex fermions counted twice) describing how the WS fermions transform around non-contractible loops on the worldsheet, and (2) sets of coefficients weighting contributions to the one-loop partition function from fermions with specific boundary conditions.

Modular invariance is a requirement for a sensible model and exists if: (1) the one-loop partition function is invariant under $S : \tau \rightarrow -1/\tau$ and $T : \tau \rightarrow \tau + 1$ transformations of the complex WS parameter τ defining the one-loop worldsheet (a torus); and (2) either a specific additional two-loop constraint is satisfied^[3] or, equivalently, the states surviving the one-loop GSO projection (GSOP) “are sensible”^[4]. Under transportation around either

of the non-contractable loops (which we denote respectively as l_α and l_β) of the one-loop worldsheet, a WS fermion $\psi^{n(n,m)}$ can undergo transformations. Consistency demands diagonalizable transformations, which are expressible purely as phase changes:

$$\psi^j \xrightarrow{l_\alpha} -\exp\{\pi i \alpha^j\} \psi^j, \quad (1.1)$$

where ψ^j represents either a real fermion ψ^n or a complex fermion $\psi^{n,m}$ and $-1 < \alpha^j \leq 1$. (For the l_β loop, β^j replaces α^j .) The boundary conditions for an unpaired real fermion ψ^n must be either periodic (Ramond) or antiperiodic (Neveu-Schwarz), *i.e.* $\alpha^n, \beta^n \in \{1, 0\}$; whereas, $\alpha^{n,m}$ and $\beta^{n,m}$ can both be rational for Weyl fermions $\psi^{n,m}$. For each loop, the set of phases α^j and β^j , respectively, define the 64-dimensional boundary vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

The contribution to the one-loop partition function, $\mathcal{Z}_{\text{fermion}}$, from the WS fermions with their chosen sets of boundary vectors, $\{\boldsymbol{\alpha}\}$ and $\{\boldsymbol{\beta}\}$, can be expressed as a weighted summation over the individual partition functions, $\mathcal{Z}(\frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}})$, for specific pairs of boundary vectors,

$$\mathcal{Z}_{\text{fermion}} = \sum_{\substack{\boldsymbol{\alpha} \in \{\boldsymbol{\alpha}\} \\ \boldsymbol{\beta} \in \{\boldsymbol{\beta}\}}} C\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right) \mathcal{Z}\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right). \quad (1.2)$$

The weights $C(\frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}})$ can be either complex or real (± 1) phases when either $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ have rational, non-integer components, but only real phases when $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are both integer vectors. One-loop modular invariance requires that $\{\boldsymbol{\alpha}\}$ and $\{\boldsymbol{\beta}\}$ be identical sets and that if $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are in $\{\boldsymbol{\alpha}\}$ then $\boldsymbol{\alpha} + \boldsymbol{\gamma}$ must be also. Thus, $\{\boldsymbol{\alpha}\}$ and $\{\boldsymbol{\beta}\}$ can be defined by a set of basis vectors (BVs) $\{\mathbf{V}_i\}$:

$$\boldsymbol{\alpha} = \sum_{i=1}^D a_i \mathbf{V}_i \pmod{2}, \quad \boldsymbol{\beta} = \sum_{i=1}^D b_i \mathbf{V}_i \pmod{2}, \quad (1.3)$$

where a_i and b_i are integers in the range 0 to $N_i - 1$, with N_i (the order of \mathbf{V}_i) the smallest positive integer such that $N_i \mathbf{V}_i = 0 \pmod{2}$ for all 64 components of \mathbf{V}_i . The vector \mathbf{V}_0 , with 1 for all components, must be present in the basis set, as required by modular invariance.

Modular invariance dictates the allowed form of the phase weights:

$$C\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right) = (-1)^{s_\alpha + s_\beta} \exp\left\{\pi i \sum_{i,j} b_i \left(k_{i,j} - \frac{1}{2} \mathbf{V}_i \cdot \mathbf{V}_j\right) a_j\right\}, \quad (1.4)$$

where s_α is the spacetime component of $\boldsymbol{\alpha}$, *etc.* (s_i is similarly used for \mathbf{V}_i below.) The rational numbers $k_{i,j}$, where $-1 < k_{i,j} \leq 1$, and BVs \mathbf{V}_i are constrained by,

$$k_{i,j} + k_{j,i} = \frac{1}{2} \mathbf{V}_i \cdot \mathbf{V}_j \pmod{2}, \quad (1.5a)$$

$$N_j k_{i,j} = 0 \pmod{2}, \quad (1.5b)$$

$$k_{i,i} + k_{i,0} = -s_i + \frac{1}{4} \mathbf{V}_i \cdot \mathbf{V}_i \pmod{2}. \quad (1.5c)$$

After appropriate integer multiplication, the dependence upon the $k_{i,j}$ can be removed to yield three direct constraints on the \mathbf{V}_i :

$$N_{i,j} \mathbf{V}_i \cdot \mathbf{V}_j = 0 \pmod{4} \quad (1.6a)$$

$$N_i \mathbf{V}_i \cdot \mathbf{V}_i = 0 \pmod{8} \quad (1.6b)$$

The number of real fermions simultaneously periodic

$$\text{for any three basis vectors is even.} \quad (1.6c)$$

$N_{i,j}$ is the lowest common multiple of N_i and N_j .

One of the non-contractible loops, l_α , may be regarded as space-like, and the other loop, l_β , as time-like. Each α corresponds to a sector of excitations of the vacuum by fermion modes at frequencies μ_{ψ^j} proportional to α^j :

$$\mu_{\psi^j} = \left\{ \frac{1 \pm \alpha^j}{2} + \mathbb{Z} \right\}, \quad (1.7)$$

(+ for ψ^j and - for ψ^{j*}). In sector α , each complex Weyl fermion $\psi^{n,m}$ carries a U(1) charge, $Q(\psi^{n,m})$, proportional to its boundary condition:

$$Q(\psi^{n,m}) = \alpha^{n,m}/2 + F(\psi^{n,m}). \quad (1.8)$$

F is the fermion number operator with eigenvalues $\{0, \pm 1\}$ for non-periodic fermions and $\{0, -1\}$ for periodic. Hence, the charge $Q(\psi^{n,m})$ has possible values of $\{0, \pm 1\}$ for antiperiodic fermions, and $\{\pm \frac{1}{2}\}$ for periodic.

The boundary vectors β contribute a set of GSOPs that act on the α -sector states, projecting some of them out of the model. Which states survive is a function of the phase coefficients $\{C(\frac{\alpha}{\beta})\}$ (or equivalently of the $\{C(\frac{\alpha}{V_i})\}$). In a given α -sector, a state is removed from the model unless it satisfies the GSOP eq. imposed by each \mathbf{V}_j :

$$\mathbf{V}_j \cdot \mathbf{F}_\alpha = \left(\sum_i k_{j,i} a_i \right) + s_j - \frac{1}{2} \mathbf{V}_j \cdot \alpha \pmod{2}, \quad (1.9a)$$

or, equivalently,

$$\mathbf{V}_j \cdot \mathbf{Q}_\alpha = \left(\sum_i k_{j,i} a_i \right) + s_j \pmod{2}. \quad (1.9b)$$

Eqs. (1.9a-b) imply that if a state with charge vector \mathbf{Q} in sector $m\alpha$ survives a given set of GSOPs, then a state with charge vector $-\mathbf{Q}$ in sector $(N_\alpha - m)\alpha$ also survives this same set. This is guaranteed because $s_j = 0, 1$ and

$$(N_i - m)S_i = -mS_i \pmod{2} \quad (1.10a)$$

$$(N_i - m)k_{j,S_i} = -mk_{j,S_i} \pmod{2}. \quad (1.10b)$$

Since $\mathbf{Q}_\alpha(\psi^{1,2})$ gives the chirality of a spacetime fermion, a model with a surviving spacetime fermion of a given chirality in sector $m\alpha$ must also have a surviving spacetime fermion of opposite chirality and opposite internal charges in sector $(N_\alpha - m)\alpha$.

2. Number of Supersymmetries in Free Fermionic Strings

2.a Classes of Basis Vectors

In D -dimensional heterotic free fermionic models, the $3(10 - D)$ real internal LM fermions (henceforth denoted by the χ^I) non-linearly realize a worldsheet supersymmetry using a supercurrent of the form^[3,4]

$$T_F = \psi^\mu \partial X_\mu + f_{IJK} \chi^I \chi^J \chi^K. \quad (2.1)$$

The f_{IJK} are the structure constants of a semi-simple Lie algebra \mathcal{L} of dimension $3(10 - D)$. Four dimensional models can involve any one of the three 18-dimensional Lie algebras: $SU(2)^6$, $SU(2) \otimes SU(4)$, and $SU(3) \otimes SO(5)$. When T_F is transported around the non-contractible loops on the worldsheet, it must transform the same as ψ^μ does, periodically for spacetime fermions and antiperiodically for spacetime bosons. This requirement severely constrains the BVs in consistent models. Each BV must represent an automorphism (up to a minus sign) of the chosen algebra, since $f_{IJK} \chi^I \chi^J \chi^K$ must also transform as ψ^μ does.

The simplest modular invariant heterotic four-dimensional string model built from free fermions is non-supersymmetric. This model contains a single BV, the all-periodic \mathbf{V}_0 , which means it has only two sectors: \mathbf{V}_0 and the all antiperiodic, $\vec{0} \equiv \mathbf{V}_0 + \mathbf{V}_0$. The graviton, dilaton, antisymmetric tensor, and spin-1 gauge particles all originate in the $\vec{0}$ sector. Each of the three choices for Lie algebra allow various possibilities for an additional BV, \mathbf{S}_i , that satisfies the automorphism constraint and can also contribute massless gravitinos. Every $\{\mathbf{V}, \mathbf{S}_i\}$ set generates an $N = 4$ supergravity model. Additional BVs (with their corresponding GSOPs) must be added to reduce the number of spacetime supersymmetries below four. Ref. [12] indicates that neither $SU(2) \otimes SU(4)$ nor $SU(3) \otimes SO(5)$ algebras can be used to obtain $N = 1$ ST-SUSY. This work also presents two examples of different BV combinations (one being the NAHE set^[13] that can yield $N = 1$ for $SU(2)^6$, while also revealing a situation where presence of a specific BV forbids $N = 1$).

Our objective is to continue the work begun in [12]. That is, we *completely* classify the sets of LM BVs that can produce exactly $N = 1$ ST-SUSY (and $N = 4, 2$, and 0 ST-SUSY solutions in the process). Therefore, we select $SU(2)^6$ for the supercurrent's Lie algebra which gives (2.1) the form of,

$$T_F = \psi^\mu \partial X_\mu + i \sum_{J=1}^6 \chi^{3J} \chi^{3J+1} \chi^{3J+2}. \quad (2.2)$$

The fermion triplet $(\chi^{3J}, \chi^{3J+1}, \chi^{3J+2})$ yields the three generators of the J^{th} $SU(2)$. The generators for each $SU(2)$ can be written using either the Cartan-Weyl (CW) basis (J_3 , J_+ , and J_-) or the non-Cartan-Weyl basis (J_3 , J_1 , and J_2).

An automorphism of $SU(2)^6$ is the product of inner automorphisms for the individual $SU(2)$ algebras and an outer automorphism of the whole $SU(2)^6$ product algebra.^[14,12] An outer automorphism can be expressed as an element of the permutation group P_6 that mixes the $SU(2)$ algebras.^[12] The elements of P_6 can be resolved into disjoint commuting cycles, and fit into eleven classes defined by the different possible lengths, n_k , of the cycles in the permutation (with a set of lengths written as $n_1 \cdot n_2 \cdots n_i$) such that $\sum_k n_k = 6$. The set of these eleven classes is

$$\mathbf{n} \in \{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1, \ 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1, \ 2 \cdot 2 \cdot 1 \cdot 1, \ 2 \cdot 2 \cdot 2, \\ 3 \cdot 1 \cdot 1 \cdot 1, \ 3 \cdot 2 \cdot 1, \ 3 \cdot 3, \ 4 \cdot 1 \cdot 1, \ 4 \cdot 2, \ 5 \cdot 1, \ 6\}. \quad (2.3)$$

The first element in this set, $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$, is the P_6 identity element, which does not permute any of the six individual $SU(2)_J$ algebras.

$$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1: \quad (\chi^{3J}, \chi^{3J+1}, \chi^{3J+2}) \leftrightarrow (\chi^{3J}, \chi^{3J+1}, \chi^{3J+2}) \text{ for } J = 1 \text{ to } 6. \quad (2.4a)$$

$2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ denotes a cyclic permutation between two $SU(2)$ algebras, with J subscripts indicating which two, *e.g.*

$$2_{1,2} \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1: \quad (\chi^3, \chi^4, \chi^5) \leftrightarrow (\chi^6, \chi^7, \chi^8). \quad (2.4b)$$

Similarly, $2 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ denotes two separate cyclic permutations between two pairs of $SU(2)$ algebras, *e.g.*

$$2_{1,2} \cdot 2_{3,4} \cdot 1 \cdot 1 \cdot 1 \cdot 1: \quad (\chi^3, \chi^4, \chi^5) \leftrightarrow (\chi^6, \chi^7, \chi^8), \\ (\chi^9, \chi^{10}, \chi^{11}) \leftrightarrow (\chi^{12}, \chi^{13}, \chi^{14}). \quad (2.4c)$$

while, as our last example, $3 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ performs a cyclic permutation between three $SU(2)$ algebras, *e.g.*

$$3_{1,2,3} \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1: \quad (\chi^3, \chi^4, \chi^5) \rightarrow (\chi^6, \chi^7, \chi^8), \\ (\chi^6, \chi^7, \chi^8) \rightarrow (\chi^9, \chi^{10}, \chi^{11}), \\ (\chi^9, \chi^{10}, \chi^{11}) \rightarrow (\chi^3, \chi^4, \chi^5). \quad (2.4d)$$

All eleven classes of cyclic permutations can be diagonalized in a new fermion basis by similarity transformations. The eigenvalue contributions from a non-disjoint cyclic permutation of length n is the set of n^{th} -roots of 1.

For spacetime fermions, diagonalized inner automorphisms of a $SU(2)$ triplet gives phases $(1, \exp\{i\pi\theta\}, \exp\{-i\pi\theta\})$, with $0 \leq \theta < 2$. For generic θ the triplet eigenstates

can only be written in the CW basis. However, for the particular values of $\theta = 0, 1$ the eigenstates can correspond to non-CW generators. Besides $(1, -1, -1)$ and $(1, 1, 1)$ for $\theta = 0, 1$ in the CW basis, the non-CW basis can include the additional eigenvalue sets of $(-1, 1, -1)$ and $(-1, -1, 1)$.

When inner and outer automorphisms are combined, the number of independent θ is reduced from one for each $(\chi^{3J}, \chi^{3J+1}, \chi^{3J+2})$ triplet to one for each non-disjoint cycle of length n_k in the outer automorphism. Table 1 shows the eigenvalues for a given cycle and θ combination. For odd n_k , the contribution to the GSOP from the fermions in the cyclic permutation is of order $2n_k$ when $\theta = 0, 1$; whereas, for even n_k , it is of order $2n_k$ when $\theta = 1$, and of order n_k when $\theta = 0$.

Massless sectors correspond to $\theta = 1$ for all non-disjoint cycles in a permutation. This means that massless sectors can be written in either the CW or the non-CW bases, depending on the basis required by massive sectors. Any other value (including 0) for any of the θ 's increases the vacuum mass of a sector. Thus, consistent models *cannot* contain spacetime fermion sectors with tachyonic vacuums, since such sectors would not correspond to automorphisms of the WS supercurrent.

Table 2 gives the α^j boundary vector components associated with the $-\exp\{\pi i \alpha^j\}$ eigenvalues in the eleven massless LM sectors \mathbf{S}_i . These sectors are formed by combining $\theta = 1$ in the inner automorphisms with a representative element from each the eleven classes of P_6 cyclic permutations listed above. Note that $\theta = 1$ for a cycle gives exactly one real periodic fermion (in that cycle). Thus, massless sectors with an even number of LM real periodic fermions must be built from permutation classes having an even number of disjoint cycles. Since the RM part of a gravitino sector is completely antiperiodic and the number of real LM periodic fermions in a gravitino-generating sector must be even by rule (1.6c), the six \mathbf{S}_i qualifying as gravitino-sources are: \mathbf{S}_1 ($1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ class), \mathbf{S}_3 ($2 \cdot 2 \cdot 1 \cdot 1$ class), \mathbf{S}_5 ($3 \cdot 1 \cdot 1 \cdot 1$ class), \mathbf{S}_7 ($3 \cdot 3$ class), \mathbf{S}_9 ($4 \cdot 2$ class), and \mathbf{S}_{10} ($5 \cdot 1$ class).^[12] These sectors are of orders $N_1 = 2$, $N_3 = 4$, $N_5 = 6$, $N_7 = 6$, $N_9 = 8$, and $N_{10} = 10$, respectively. Models formed simply from the periodic sector \mathbf{V}_0 and any one of these six \mathbf{S}_i yield $N = 4$ ST-SUSY after GSOPs. The four gravitinos are dispersed among the sectors $m\mathbf{S}_i$, where $m = 1, 3, 5, \dots, N_i - 1$. (Therefore, only for \mathbf{S}_1 do all four gravitinos appear in the same sector.)

2.b General Requirements of $N = 1$ Solutions

If any one of the six gravitino-generating \mathbf{S}_i is to produce $N = 1$ ST-SUSY, additional BVs giving satisfactory GSOPs must be added to the initial $\{\mathbf{V}_0, \mathbf{S}_i\}$ set. (We often refer to these additional BVs as reduction vectors.) Focusing on $(N = 1)$ supersymmetric free fermionic models offers one obvious simplification for model classification: whether a basis vector is a spacetime fermion or boson is of no physical consequence. Therefore, we can

choose that all BV be spacetime fermions, which is of use in analysis of GSOPs. Another non-physical degree of freedom in $N = 1$ models is the choice of which sector $m\mathbf{S}_i$ (of order N_{S_i}) should produce the one surviving gravitino. We choose the LM gravitino to always appear in \mathbf{S}_i . Our discussion following eq. (1.9a-b) indicates that if a gravitino of given chirality and charge vector \mathbf{Q} in sector $m\mathbf{S}_i$ survives a set of GSOPs, a gravitino of opposite chirality and charge vector $-\mathbf{Q}$ in sector $(N_{S_i} - m)\mathbf{S}_i$ survives also. Hence, $N = 1$ ST-SUSY implies that the set of GSOPs from the reduction vectors must act together to form a chiral operator that projects out all gravitinos from \mathbf{S}_i (for $i \neq 1$) except one left-handed. Then only a single right-handed gravitino remains in $(N_{S_i} - 1)\mathbf{S}_1$.

For a consistent model, the additional BVs must be derived from automorphisms that commute among themselves and with \mathbf{S}_i . This requires the various $SU(2)^6$ permutations to commute, necessitating a disjoint cycle or product of disjoint cycles in one BV to be a power or “root” of disjoint cycles in all other BVs.^[12] For example, \mathbf{S}_5 uses the permutation $3_{1,2,3} \cdot 1_4 \cdot 1_5 \cdot 1_6$. Together the identity transformations, $1_4 \cdot 1_5 \cdot 1_6$, for the last three $SU(2)$ algebras, correspond to the cube of the $3_{4,5,6}$ permutation and to the square of the $2_{4,5} \cdot 1_6$ permutation. Similarly, $3_{1,2,3}$ commutes only with itself, with its square $3_{1,3,2}$, and with its cube $1_1 \cdot 1_2 \cdot 1_3$. Since $3_{4,5,6}$ and $2_{4,5} \cdot 1_6$ do not commute, there are two (unique classes of) permutation sets we can use to form the reduction vectors for \mathbf{S}_5 :

$$\{3_{1,2,3} \cdot 1_4 \cdot 1_5 \cdot 1_6, 3_{1,2,3} \cdot 3_{4,5,6}, 1_1 \cdot 1_2 \cdot 1_3 \cdot 3_{4,5,6}, 1_1 \cdot 1_2 \cdot 1_3 \cdot 1_4 \cdot 1_5 \cdot 1_6\}, \quad (2.5)$$

and

$$\{3_{1,2,3} \cdot 1_4 \cdot 1_5 \cdot 1_6, 3_{1,2,3} \cdot 2_{4,5} \cdot 1_6, 1_1 \cdot 1_2 \cdot 1_3 \cdot 2_{4,5} \cdot 1_6, 1_1 \cdot 1_2 \cdot 1_3 \cdot 1_4 \cdot 1_5 \cdot 1_6\}. \quad (2.6)$$

The potential set of reduction vectors is formed by combining the allowed cyclic permutations with inner automorphisms.

Generically, $N = 1$ ST-SUSY requires the non-CW basis with $\theta = 0, 1$ for inner automorphisms.^[12] Therefore, the inner automorphisms that may be combined with a given set of permutations are $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$, and $(1, 1, 1)$ for spacetime fermions. One constraint on inner automorphisms occurs when a product of cycles of equal length is a power of a single larger cycle also appearing, for example, when $1_4 \cdot 1_5 \cdot 1_6$ and $3_{4,5,6}$ are both used. In this event, all cycles in the product must be tensored with the same inner automorphism. A second constraint is that LM BVs must satisfy (1.6a-c) when \mathbf{S}_i is one of the \mathbf{V}_i .

In Table 3, we find all boundary vectors formed from powers of a non-disjoint cyclic permutation with length n_k in the range $1 \leq n_k \leq 6$ with $\theta = 1, 0$. These form maximal sets of commuting boundary vectors of length n_k . Boundary vectors that could be used to construct the unique massless gravitino generators are found. The independent classes of GSOPs are then determined for these specific boundary vectors. We use these results in

the next subsection to form all possible sets of reduction vectors and their related GSOPs on each of the six gravitino generators. For a given gravitino generator we classify the reduction sets by specifying the longest length independent cycles appearing in a reduction set. If we include the gravitino generator in its own reduction set, then there are 11 classes of models for \mathbf{S}_1 , four classes for \mathbf{S}_3 , two classes for \mathbf{S}_5 , two for \mathbf{S}_7 , and one each for \mathbf{S}_9 and \mathbf{S}_{10} . We analyze the sets of GSOPs to determine which (if any) are sufficient to produce *exactly* $N = 1$ ST-SUSY from a given gravitino generator.

2.c Classification of Supersymmetry Solutions

In this subsection we present the details of our research. We investigate all possible combinations of unique GSOPs imposed by the BVs in reduction sets for each of the six distinct gravitino generators. The BVs are formed by tensoring various $S(n_k, y, z, w)$ boundary vectors together with a periodic spacetime component (denoted by “(st)”),

$$\mathbf{V}_i = (\text{st}) \prod_k S(n_k, y, z, w) , \quad (2.7)$$

such that $\sum_k n_k = 6$. (By convention we use the ordering $n_{k_1} \geq n_{k_2}$ if $k_1 < k_2$.) Thus, only the periodic spacetime component can be combined with an $S(6, y, z, w)$, whereas the various $S(5, y, z, w)$ must also be combined with the $S(1, 1, 1, w')$. In the latter case, we require $w, w' \in \{1, 4\}$ or $w, w' \in \{2, 3\}$. This enables the two real periodic J_3 fermions in $S(5, y, z, w)$ and $S(1, 1, 1, w')$ to form a complex fermion, as is required by (1.6c) for the related choices of \mathbf{S}_1 and \mathbf{S}_{10} in class X below. There are similar constraints for tensor products of the $S(n_k, y, z, w)$ in the other classes also.

The GSOPs from two generic BVs, \mathbf{V}_1 and \mathbf{V}_2 , in gravitino generator \mathbf{S}_i 's reduction set are in the same projection class if (1) their \mathbf{F} coefficients all match or differ only by an overall sign factor, and (2) the difference between the dot products of \mathbf{V}_1 and \mathbf{V}_2 with \mathbf{S}_i is $0 \pmod{\frac{4}{N_{S_i}}}$. When two GSOPs are in the same class they either work identically on the gravitino sector or in combination kick out all gravitinos. Since we desire $N = 1$ ST-SUSY, we obviously do not want all gravitinos removed. Therefore, once we choose the K_{V_1, S_i} in \mathbf{V}_1 's GSOP, K_{V_2, S_i} is fixed. For a given reduction set we present one element from each class of GSOPs. However, the classes of GSOPs for a given reduction set are not all independent. For example, when $\mathbf{S}_1 = (\text{st})S(4, 4, 1, 2)S(2, 2, 1, 2)$ the left-hand sides of the fifth through tenth GSOPs are linear combinations of those of the first four GSOPs. (See below.) Thus, the third general $N = 1$ solution for this \mathbf{S}_1 expression is not independent of the first two.

We define \mathbf{F}_0 as the number operator for periodic ψ^μ and $\mathbf{F}_{1,2,3}$ for the (up to) three internal periodic WS fermions. Recall that $\mathbf{F}_i \in \{0, -1\}$ for periodic fermions. In our GSOPs below, we replace the right-hand side (RHS) of (1.9a) with the corresponding set of possible RHS eigenvalues. For a gravitino state of given chirality and internal charge

vector \mathbf{Q} to survive a GSOP, the left-hand side (LHS) of the GSOP equation must equal one of the possible values of the RHS, otherwise the gravitino state is projected out of the model.

For any gravitino generator \mathbf{S}_i , the first GSOP (with all “+1” \mathbf{F} coefficients) is always present, since \mathbf{S}_i and the periodic sector \mathbf{V}_0 are both in the class of BVs giving this projection. Application of this GSOP to \mathbf{S}_i and application of the corresponding all-integer component GSOPs to the odd multiples of \mathbf{S}_i always produces $N = 4$ ST-SUSY. This is the starting point for our models below. Hence, we generally do not discuss the first GSOP of any set.

Several LM BVs that generate GSOPs in our sets must be paired up with RM BVs of sufficient order to satisfy (1.6a-b). Otherwise their GSOPs could not be present. GSOP equations resulting from BVs of this type are denoted by an “*” in front of the GSOP class name. Most often such projections result directly in $N = 0$ ST-SUSY. Note, however, that GSOPs from some LM BVs that do not require specific RM pairing can also automatically eliminate ST-SUSY. All such GSOPs at odds with $N > 0$ are marked with a “ $\rightarrow N = 0$ ” before their class name.

XI. LONGEST CYCLIC PERMUTATIONS: 6 in $S(6, 1, 1, 1)$

ALLOWED GRAVITINO GENERATORS: $\mathbf{S}_1, \mathbf{S}_7$

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POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(6, 6, 1, 2)$:

$1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\}$	class (1.6.2.1)
$1 F_0 + \frac{1}{3} F_1 + 1 F_2 - \frac{1}{3} F_3 \in \{0, 1\}$	class (1.6.2.3, 4)
$1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0$ * class (1.6.2.2)
$1 F_0 + \frac{2}{3} F_1 + 0 F_2 - \frac{2}{3} F_3 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0$ * class (1.6.2.5, 6)
$1 F_0 + \frac{5}{6} F_1 + \frac{1}{2} F_2 + \frac{1}{6} F_3 \in \{-\frac{1}{4}, \frac{3}{4}\}$	$\rightarrow N = 0$ * class (1.6.2.7, 8)
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + \frac{1}{2} F_3 \in \{\frac{1}{4}, -\frac{3}{4}\}$	$\rightarrow N = 0$ * class (1.6.2.9, 10)
$1 F_0 + \frac{1}{6} F_1 + \frac{1}{2} F_2 + \frac{5}{6} F_3 \in \{-\frac{1}{4}, \frac{3}{4}\}$	$\rightarrow N = 0$ * class (1.6.2.9, 10)

The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$ or $N = 0$.

.....
POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_7 = m \times (\text{st})S(6, 2, 1, 2)$:

$1 F_0 + 1 F_1 \in \{0, \pm \frac{1}{3}, \pm \frac{2}{3}, 1\}$	class (3.6.2.1)
$1 F_0 + 0 F_1 \in \{\pm \frac{1}{6}, \pm \frac{1}{2} \pm \frac{5}{6}\}$	$\rightarrow N = 0$ * class (3.6.2.2)
$1 F_0 + \frac{1}{2} F_1 \in \{-\frac{1}{12}, \frac{1}{4}, -\frac{5}{12}, \frac{7}{12}, -\frac{3}{4}, \frac{11}{12}\}$	$\rightarrow N = 0$ * class (3.6.2.3, 4)

For \mathbf{S}_7 the gravitinos are divided up between \mathbf{S}_7 , $3\mathbf{S}_7$, and $5\mathbf{S}_7$. However, as shown in section 1, states in $\{m\mathbf{S}_7\}$ are not independent. For every gravitino of given chirality and charge \mathbf{Q}_1 in \mathbf{S}_7 , there is a gravitino of opposite chirality and charge $-\mathbf{Q}_1$ in $5\mathbf{S}_7$. Thus, we need only examine the GSOPs for \mathbf{S}_7 and $3\mathbf{S}_7$. For $3\mathbf{S}_7$ we make use of the $K_{i,3j} = 3K_{i,j}$ identity and of the equivalence of $3\mathbf{S}_7$ from $3(\text{st})S(6, 2, 1, 2)$ with \mathbf{S}_1 from $(\text{st})S(6, 6, 1, 2)$. (In general, if $N_i/2$ for \mathbf{S}_i is odd, then $\frac{N_i}{2}\mathbf{S}_i$ equals \mathbf{S}_1 .) \mathbf{S}_7 can generate $N = 1$ SUSY only if all gravitinos are removed from $3\mathbf{S}_7$.

The class (3.6.2.1) GSOP for \mathbf{S}_7 originates from the same BVs that generate classes (1.6.2.1) and (1.6.2.3, 4) for $3\mathbf{S}_7$, while together the BVs for (3.6.2.2) and (3.6.2.3, 4) compose the sets associated with all remaining GSOP classes for $3\mathbf{S}_7$. If they are present, the GSOPs in (3.6.2.2) and (3.6.2.3, 4) and the related in (1.6.2.2, 5–10) remove all gravitinos from \mathbf{S}_7 and $3\mathbf{S}_7$, respectively, thereby resulting in $N = 0$ ST-SUSY. GSO class (3.6.2.1) either keeps one gravitino of each chirality (RHS equals 0 or 1) or no gravitinos at all (RHS equals $\pm\frac{2}{3}$) in \mathbf{S}_7 . A RHS from the set $\{0, \pm\frac{2}{3}\}$ in GSOP (3.6.2.1) implies a RHS of 1 for (1.6.2.1) and a RHS of 0 for (1.6.2.3, 4); a RHS from the set $\{1, \pm\frac{1}{3}\}$ in GSOP (3.6.2.1) implies a RHS of 0 for (1.6.2.1) and a RHS of 1 for (1.6.2.3, 4). Thus, when one of each chirality is kept in \mathbf{S}_7 , two of each chirality automatically survive in $3\mathbf{S}_7$, resulting in $N = 4$ ST-SUSY. However, if no gravitinos are kept in \mathbf{S}_7 , either two or zero gravitinos remain in $3\mathbf{S}_7$.

The same patterns occurs for the three other sets of BVs in Table 3 that commute with $S(6, 1, 1, 1)$. Consequently, any model using non-disjoint cyclical permutations of length six (equivalently, \mathbb{Z}_{12} twists) can never possess $N = 1$ ST-SUSY.

X. LONGEST CYCLIC PERMUTATIONS: $5 \cdot 1$ in $S(5, 1, 1, 1)S(1, 1, 1, 1)$

ALLOWED GRAVITINO GENERATORS: $\mathbf{S}_1, \mathbf{S}_{10}$

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POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(5, 5, 1, 1)S(1, 1, 1, 1)$:

$1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\}$	class (1.5.1.1)
$1 F_0 + \frac{3}{5} F_1 - \frac{1}{5} F_2 + 1 F_3 \in \{-\frac{1}{5}, \frac{4}{5}\}$	$\rightarrow N = 0$ class (1.5.1.3, 4)
$1 F_0 + \frac{1}{5} F_1 + \frac{3}{5} F_2 + 1 F_3 \in \{-\frac{2}{5}, \frac{3}{5}\}$	$\rightarrow N = 0$ class (1.5.1.5, 6)
$1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0$ * class (1.5.1.2)
$1 F_0 + \frac{4}{5} F_1 + \frac{2}{5} F_2 + 0 F_3 \in \{-\frac{1}{10}, \frac{9}{10}\}$	$\rightarrow N = 0$ * class (1.5.1.7, 8)
$1 F_0 + \frac{2}{5} F_1 - \frac{4}{5} F_2 + 0 F_3 \in \{-\frac{3}{10}, \frac{7}{10}\}$	$\rightarrow N = 0$ * class (1.5.1.9, 10)

The initial $N = 4$ ST-SUSY can only be broken to $N = 0$.

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_{10} = m \times (\text{st})S(5, 1, 1, 1)S(1, 1, 1, 1)$:

$$\begin{aligned}
1 F_0 + 1 F_1 &\in \{0, \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{3}{5}, \pm \frac{4}{5}, 1\} && \text{class (5.5.1.1)} \\
1 F_0 + 0 F_1 &\in \{\pm \frac{1}{10}, \pm \frac{3}{10}, \pm \frac{5}{10}, \pm \frac{7}{10}, \pm \frac{9}{10}\} && \rightarrow N = 0 * \text{class (5.5.1.2)}
\end{aligned}$$

Since \mathbf{S}_{10} generates gravitinos in each of its odd multiples, we need to consider the interdependent sets of GSOPs on \mathbf{S}_{10} , $3\mathbf{S}_{10}$, and $5\mathbf{S}_{10}$. (5.5.1.1) and (5.5.1.2) form the set of GSOPs for both \mathbf{S}_{10} and $3\mathbf{S}_{10}$. A RHS of 0 (1) in (5.5.1.1) for \mathbf{S}_{10} leads to a RHS of 1 (0) for $3\mathbf{S}_{10}$. This results in one gravitino of each chirality surviving in both \mathbf{S}_{10} and $3\mathbf{S}_{10}$. Any other RHS value for (5.5.1.1) for \mathbf{S}_{10} removes all gravitinos from both sectors.

No gravitinos can emerge from $5\mathbf{S}_{10}$. Since $5\mathbf{S}_{10}$ equals \mathbf{S}_1 , GSOP classes (1.5.1.1-10) apply to $5\mathbf{S}_{10}$ as well. Eq. (1.5c) relates the GSOPs in class (1.5.1.1) coming from \mathbf{V}_0 and \mathbf{S}_{10} :

$$K_{S_{10}, 5S_{10}} = 5K_{S_{10}, S_{10}} = 5K_{V_0, S_{10}} + 1 \pmod{2}. \quad (2.8)$$

Hence, \mathbf{S}_{10} and \mathbf{V}_0 contribute conflicting versions of GSOP (1.5.1.1) for \mathbf{S}_{10} that contain differing RHS's: $5K_{V_0, S_{10}} + \frac{3}{5} \pmod{2}$ and $5K_{V_0, S_{10}} + 1 \pmod{2}$, respectively. Therefore, $\{m\mathbf{S}_{10}\}$ generates either an $N = 4$ or $N = 0$ ST-SUSY.

IX. LONGEST CYCLIC PERMUTATIONS: $4 \cdot 2$ in $S(4, 1, 1, 1)S(2, 1, 1, 1)$

ALLOWED GRAVITINO GENERATORS: \mathbf{S}_1 , \mathbf{S}_3 , and \mathbf{S}_9 .

POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(4, 4, 1, 1)S(2, 2, 1, 1)$:

$$\begin{aligned}
1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 &\in \{0, 1\} && \text{class (1.4.1.1)(1.2.1.1)} \\
1 F_0 + 1 F_1 + 0 F_2 + 0 F_3 &\in \{0, 1\} && \text{class (1.4.1.2)(1.2.1.4)} \\
1 F_0 + 0 F_1 + 1 F_2 + 1 F_3 &\in \{\pm \frac{1}{2}\} && \rightarrow N = 0 \quad \text{class (1.4.1.3)(1.2.1.1)} \\
1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 &\in \{\pm \frac{1}{2}\} && \rightarrow N = 0 * \text{class (1.4.1.4)(1.2.1.4)} \\
1 F_0 + \frac{1}{2} F_1 + 1 F_2 + 0 F_3 &\in \{-\frac{1}{4}, +\frac{3}{4}\} && \rightarrow N = 0 \quad \text{class (1.4.1.5, 6)(1.2.1.2)} \\
1 F_0 + \frac{1}{2} F_1 + 0 F_2 + 1 F_3 &\in \{-\frac{1}{4}, +\frac{3}{4}\} && \rightarrow N = 0 \quad \text{class (1.4.1.7, 8)(1.2.1.3)}
\end{aligned}$$

The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$.

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 POSSIBLE GSO PROJECTIONS ON $\mathcal{S}_1 = (\text{st})S(4, 4, 1, 2)S(2, 2, 1, 2)$:

$1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\}$	class (1.4.2.1)(1.2.2.1)
$1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 \in \{0, 1\}$	class (1.4.2.2)(1.2.2.1)
$1 F_0 + \frac{3}{4} F_1 + \frac{1}{4} F_2 + 0 F_3 \in \{0, 1\}$	class (1.4.2.3, 4)(1.2.2.2)
$1 F_0 + \frac{3}{4} F_1 + \frac{1}{4} F_2 + \frac{1}{2} F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\}$	class (1.4.2.w)(1.2.2.w) w/ $w = 3, 4$
$1 F_0 + \frac{3}{4} F_1 + \frac{1}{4} F_2 - \frac{1}{2} F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\}$	class (1.4.2.w)(1.2.2.w') w/ $(w, w') = (3, 4), (4, 3)$
$1 F_0 + \frac{1}{4} F_1 + \frac{3}{4} F_2 + 0 F_3 \in \{0, 1\}$	class (1.4.2.7, 8)(1.2.2.2)
$1 F_0 + \frac{1}{4} F_1 + \frac{3}{4} F_2 + \frac{1}{2} F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\}$	class (1.4.2.w)(1.2.2.w') w/ $(w, w') = (7, 3), (8, 4)$
$1 F_0 + \frac{1}{4} F_1 + \frac{3}{4} F_2 - \frac{1}{2} F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\}$	class (1.4.2.w)(1.2.2.w') w/ $(w, w') = (7, 4), (8, 3)$
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + 1 F_3 \in \{0, 1\}$	class (1.4.2.5, 6)(1.2.2.1)
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + 0 F_3 \in \{\pm\frac{1}{2}\}$	class (1.4.2.5, 6)(1.2.2.2)
$1 F_0 + 1 F_1 + 1 F_2 + 0 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class (1.4.2.1)(1.2.2.2)}$
$1 F_0 + 1 F_1 + 1 F_2 + \frac{1}{2} F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\}$	$\rightarrow N = 0 * \text{class (1.4.2.1)(1.2.2.3, 4)}$
$1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class (1.4.2.2)(1.2.2.2)}$
$1 F_0 + 0 F_1 + 0 F_2 + \frac{1}{2} F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class (1.4.2.2)(1.2.2.3, 4)}$
$1 F_0 + \frac{3}{4} F_1 + \frac{1}{4} F_2 + 1 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class (1.4.2.3, 4)(1.2.2.1)}$
$1 F_0 + \frac{1}{4} F_1 + \frac{3}{4} F_2 + 1 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class (1.4.2.7, 8)(1.2.2.1)}$
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + \frac{1}{2} F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\}$	$\rightarrow N = 0 * \text{class (1.4.2.w)(1.2.2.w')}$ w/ $(w, w') = (5, 3), (6, 4)$
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 - \frac{1}{2} F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\}$	$\rightarrow N = 0 * \text{class (1.4.2.w)(1.2.2.w')}$ w/ $(w, w') = (5, 4), (6, 3)$

There are three general solutions for $N = 1$ ST-SUSY. GSOP class (1.4.2.1)(1.2.2.1) can be combined with:

1. (1.4.2.3, 4)(1.2.2.2) and/or (1.4.2.7, 8)(1.2.2.2) plus, optionally, (1.4.2.2)(1.2.2.1) and/or (1.4.2.5, 6)(1.2.2.1),
2. (1.4.2.3)(1.2.2.3)/(1.4.2.4)(1.2.2.4) and/or (1.4.2.7)(1.2.2.4)/(1.4.2.8)(1.2.2.3), plus, optionally, (1.4.2.2)(1.2.2.1) and/or (1.4.2.5, 6)(1.2.2.2), or
3. (1.4.2.3)(1.2.2.4)/(1.4.2.4)(1.2.2.3) and/or (1.4.2.7)(1.2.2.3)/(1.4.2.8)(1.2.2.4), plus, optionally, (1.4.2.2)(1.2.2.1) and/or (1.4.2.5, 6)(1.2.2.2).

For each of these three $N = 1$ solution sets, choice of the RHS value for either of the first two GSOPs fixes those for all other GSOs used.

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_3 = m \times (\text{st})S(4, 2, 1, 1)S(2, 2, 1, 1)$:

$1 F_0 + 1 F_1 + 1 F_2 \in \{0, \pm\frac{1}{2}, 1\}$	class (2.4.1.1)(1.2.1.1)
$1 F_0 + 0 F_1 + 0 F_2 \in \{0, \pm\frac{1}{2}, 1\}$	class (2.4.1.4)(1.2.1.4)
$1 F_0 + 1 F_1 + 0 F_2 \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\}$	$\rightarrow N = 0$ class (2.4.1.2)(1.2.1.2)
$1 F_0 + 0 F_1 + 1 F_2 \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\}$	$\rightarrow N = 0$ class (2.4.1.3)(1.2.1.3)

The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$ or $N = 0$.

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_3 = m \times (\text{st})S(4, 2, 1, 2)S(2, 2, 1, 2)$:

$1 F_0 + 1 F_1 + 1 F_2 \in \{0, \pm\frac{1}{2}, 1\}$	class (2.4.2.1)(1.2.2.1)
$1 F_0 + 1 F_1 + 0 F_2 \in \{0, \pm\frac{1}{2}, 1\}$	* class (2.4.2.1)(1.2.2.2)
$1 F_0 + 0 F_1 + 1 F_2 \in \{0, \pm\frac{1}{2}, 1\}$	class (2.4.2.2)(1.2.2.1)
$1 F_0 + 0 F_1 + 0 F_2 \in \{0, \pm\frac{1}{2}, 1\}$	* class (2.4.2.2)(1.2.2.2)
$1 F_0 + \frac{1}{2} F_1 + \frac{1}{2} F_2 \in \{0, \pm\frac{1}{2}, 1\}$	class (2.4.2.w)(1.2.2.w), w/ $w = 3, 4$
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 \in \{0, \pm\frac{1}{2}, 1\}$	class (2.4.2.w)(1.2.2.w'), w/ $(w, w') = (3, 4), (4, 3)$
$1 F_0 + 1 F_1 + \frac{1}{2} F_2 \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\}$	$\rightarrow N = 0$ * class (2.4.2.1)(1.2.2.3, 4)
$1 F_0 + 0 F_1 + \frac{1}{2} F_2 \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\}$	$\rightarrow N = 0$ * class (2.4.2.2)(1.2.2.3, 4)
$1 F_0 + \frac{1}{2} F_1 + 1 F_2 \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\}$	$\rightarrow N = 0$ class (2.4.2.3, 4)(1.2.2.1)
$1 F_0 + \frac{1}{2} F_1 + 0 F_2 \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\}$	$\rightarrow N = 0$ class (2.4.2.3, 4)(1.2.2.2)

$N = 1$ ST-SUSY is directly reached through proper choice of RHS values of (1) the fifth or sixth GSOP, or (2) any two of the second through fourth GSOPs. In either case, the RHS values of the rest of the first six GSOPs also used are fixed or $N = 1$ is broken to $N = 0$. The first four GSOPs imitate \mathbf{S}_1 's NAHE set of projections in I.

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_9 = m \times (\text{st})S(4, 1, 1, 1)S(2, 1, 1, 1)$:

$1 F_0 + 1 F_1 \in \{0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, 1\}$	class (4.4.1.w)(2.2.1.w), w/ $w = 1, 2$
$1 F_0 + 1 F_1 \in \{0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, 1\}$	class (4.4.1.w)(2.2.1.w'), w/ $(w, w') = (1, 2), (2, 1)$
$1 F_0 + 0 F_1 \in \{0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, 1\}$	class (4.4.1.w)(2.2.1.w), w/ $w = 3, 4$
$1 F_0 + 0 F_1 \in \{0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, 1\}$	class (4.4.1.w)(2.2.1.w'),

$$w / (w, w') = (3, 4), (4, 3)$$

For \mathbf{S}_9 the gravitinos are divided among $m\mathbf{S}_9$, $m = 1, 3, 5, 7$. \mathbf{S}_9 and $3\mathbf{S}_9$ have the same set of projections classes, with the RHS of $3\mathbf{S}_9$'s projections dependent upon those for \mathbf{S}_9 . The first GSOP keeps one left-handed and one right-handed gravitino in \mathbf{S}_9 and also in $3\mathbf{S}_9$. The internal charge vector \mathbf{Q} on surviving gravitinos of given chirality changes sign between \mathbf{S}_9 and $3\mathbf{S}_9$ for this GSOP. However, the second GSOP operator requires these charge vectors to be equal. Therefore, when both the first and second GSOPs are applied all gravitinos in $3\mathbf{S}_9$ can be projected out. This yields $N = 2$ ST-SUSY from \mathbf{S}_9 . The last two projections eliminate one chirality for \mathbf{S}_9 gravitinos, therefore inducing $N = 1$.

VIII. LONGEST CYCLIC PERMUTATIONS: $4 \cdot 1 \cdot 1$ in $S(4, 1, 1, 1)S(1, 1, 1, 1)^2$

ALLOWED GRAVITINO GENERATORS: $\mathbf{S}_1, \mathbf{S}_3$

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POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(4, 4, 1, 1)S(1, 1, 1, 1)^2$:

$1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\}$		class (1.4.1.1)(1.1.1.1)(1.1.1.1)
$1 F_0 + 1 F_1 + 0 F_2 + 0 F_3 \in \{0, 1\}$		class (1.4.1.2)(1.1.1.2)(1.1.1.2)
$1 F_0 + 0 F_1 + 1 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0$	class (1.4.1.3)(1.1.1.1)(1.1.1.1)
$1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0$	* class (1.4.1.4)(1.1.1.2)(1.1.1.2)
$1 F_0 + \frac{1}{2} F_1 + 1 F_2 + 0 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\}$	$\rightarrow N = 0$	class (1.4.1.5, 6)(1.1.1.1)(1.1.1.2)
$1 F_0 + \frac{1}{2} F_1 + 0 F_2 + 1 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\}$	$\rightarrow N = 0$	class (1.4.1.7, 8)(1.1.1.2)(1.1.1.1)

These GSOPs are the same as the set for $\mathbf{S}_1 = (\text{st})S(4, 4, 1, 1)S(2, 2, 1, 1)$ in IX. We replace (1.2.1.1) with $(1.1.1.1)^2$, (1.2.1.2) with $(1.1.1.1)(1.1.1.2)$, (1.2.1.3) with $(1.1.1.2)(1.1.1.1)$, and (1.2.1.4) with $(1.1.1.2)^2$. The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$.

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POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(4, 4, 1, 2)S(1, 1, 1, 1)^2$:

$1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\}$		class (1.4.2.1)(1.1.1.1) ²
$1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 \in \{0, 1\}$		class (1.4.2.2)(1.1.1.1) ²
$1 F_0 + \frac{3}{4} F_1 + \frac{1}{4} F_2 + 0 F_3 \in \{0, 1\}$		class (1.4.2.3, 4)(1.1.1.2) ²
$1 F_0 + \frac{1}{4} F_1 + \frac{3}{4} F_2 + 0 F_3 \in \{0, 1\}$		class (1.4.2.7, 8)(1.1.1.2) ²
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + 1 F_3 \in \{0, 1\}$		class (1.4.2.5, 6)(1.1.1.1) ²
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + 0 F_3 \in \{\pm \frac{1}{2}\}$		class (1.4.2.5, 6)(1.1.1.2) ²
$1 F_0 + 1 F_1 + 1 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0$	* class (1.4.2.1)(1.1.1.2) ²
$1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0$	* class (1.4.2.2)(1.1.1.2) ²
$1 F_0 + \frac{3}{4} F_1 + \frac{1}{4} F_2 + 1 F_3 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0$	* class (1.4.2.3, 4)(1.1.1.1) ²

$$1 F_0 + \frac{1}{4} F_1 + \frac{3}{4} F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} \quad \rightarrow N = 0 * \text{class } (1.4.2.7, 8)(1.1.1.1)^2$$

These GSOPs are a subset of those for $\mathbf{S}_1 = S(4, 4, 1, 2)S(2, 2, 1, 2)$ in IX. We replace (2.2.1.1) with $(1.1.1.1)^2$, (2.2.1.2) with $(1.1.1.2)^2$, and remove the GSOPs involving (2.2.1.3) and (2.2.1.4). There remains a single general solution for $N = 1$ ST-SUSY. (1.4.2.1)(1.1.1.1)² can be combined:

1. (1.4.2.3, 4)(1.1.1.2)² and/or (1.4.2.7, 8)(1.1.1.2)² plus, optionally, (1.4.2.2)(1.1.1.1)² and/or (1.4.2.5, 6)(1.1.1.1)².

Choice of the RHS value for either of the first two GSOPs in this $N = 1$ solution set fixes those for all other GSOs used from the set.

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_3 = m \times (\text{st})S(4, 2, 1, 1)S(1, 1, 1, 1)^2$:

$$\begin{array}{ll} 1 F_0 + 1 F_1 + 1 F_2 \in \{0, \pm \frac{1}{2}, 1\} & \text{class } (2.4.1.1)(1.1.1.1)^2 \\ 1 F_0 + 0 F_1 + 0 F_2 \in \{0, \pm \frac{1}{2}, 1\} & \text{class } (2.4.1.4)(1.1.1.2)^2 \\ 1 F_0 + 1 F_1 + 0 F_2 \in \{\pm \frac{1}{4}, \pm \frac{3}{4}\} & \rightarrow N = 0 \quad \text{class } (2.4.1.2)(1.1.1.1)(1.1.1.2) \\ 1 F_0 + 0 F_1 + 1 F_2 \in \{\pm \frac{1}{4}, \pm \frac{3}{4}\} & \rightarrow N = 0 \quad \text{class } (2.4.1.3)(1.1.1.2)(1.1.1.1) \end{array}$$

These GSOPs are the same as the set for $\mathbf{S}_3 = (\text{st})S(4, 2, 1, 1)S(2, 2, 1, 1)$. We simply replace (1.2.1.1) with $(1.1.1.1)^2$, (1.2.1.2) with $(1.1.1.1)(1.1.1.2)$, (1.2.1.3) with $(1.1.1.2)$ $(1.1.1.1)$, and (1.2.1.4) with $(1.1.1.2)^2$. The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$.

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_3 = m \times (\text{st})S(4, 2, 1, 2)S(1, 1, 1, 1)^2$:

$$\begin{array}{ll} 1 F_0 + 1 F_1 + 1 F_2 \in \{0, \pm \frac{1}{2}, 1\} & \text{class } (2.4.2.1)(1.1.1.1)^2 \\ 1 F_0 + 1 F_1 + 0 F_2 \in \{0, \pm \frac{1}{2}, 1\} & * \text{class } (2.4.2.1)(1.1.1.2)^2 \\ 1 F_0 + 0 F_1 + 1 F_2 \in \{0, \pm \frac{1}{2}, 1\} & \text{class } (2.4.2.2)(1.1.1.1)^2 \\ 1 F_0 + 0 F_1 + 0 F_2 \in \{0, \pm \frac{1}{2}, 1\} & * \text{class } (2.4.2.2)(1.1.1.2)^2 \\ 1 F_0 + \frac{1}{2} F_1 + 1 F_2 \in \{\pm \frac{1}{4}, \pm \frac{3}{4}\} & \rightarrow N = 0 \quad \text{class } (2.4.2.3, 4)(1.1.1.1)^2 \\ 1 F_0 + \frac{1}{2} F_1 + 0 F_2 \in \{\pm \frac{1}{4}, \pm \frac{3}{4}\} & \rightarrow N = 0 \quad \text{class } (2.4.2.3, 4)(1.1.1.2)^2 \end{array}$$

These GSOPs are a subset of those for $\mathbf{S}_3 = S(4, 2, 1, 1)S(2, 2, 1, 2)$ in IX. We replace (1.2.2.1) with $(1.1.1.1)^2$ and (1.2.2.2) with $(1.1.1.2)^2$, and remove the GSOPs involving (1.2.2.3) and (1.2.2.4). The first four GSOPs imitate the NAHE set. The combination of any three of these four GSOPs produce $N = 1$ ST-SUSY independent of their RHS values. The remaining GSOP either keeps or breaks $N = 1$, depending upon its RHS value. However, we cannot consider this as a category VIII solution; it actually belongs in category III (below). The first four GSOP classes are not associated with $4 \cdot 1 \cdot 1$ cycles, instead only with cycles of lengths $2 \cdot 2 \cdot 1 \cdot 1$ and $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$.

VII. LONGEST CYCLIC PERMUTATIONS: $3 \cdot 3$ in $S(3, 1, 1, 1)S(3, 1, 1, 1)$

ALLOWED GRAVITINO GENERATORS: $\mathbf{S}_1, \mathbf{S}_5, \mathbf{S}_7$

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POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(3, 3, 1, 1)^2$:

$1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\}$	class (1.3.1.1)(1.3.1.1)
$1 F_0 + \frac{1}{3} F_1 + \frac{1}{3} F_2 + 1 F_3 \in \{0, 1\}$	class (1.3.1.w)(1.3.1.w)
	w/ $w = 3, 4$
$1 F_0 + \frac{1}{3} F_1 - \frac{1}{3} F_2 + 1 F_3 \in \{0, 1\}$	class (1.3.1.w)(1.3.1.w')
	w/ $(w, w') = (3, 4), (4, 3)$
$1 F_0 + 1 F_1 + \frac{1}{3} F_2 + 1 F_3 \in \{+\frac{1}{3}, -\frac{2}{3}\}$	$\rightarrow N = 0$ class (1.3.1.1)(1.3.1.3, 4)
$1 F_0 + \frac{1}{3} F_1 + 1 F_2 + 1 F_3 \in \{+\frac{1}{3}, -\frac{2}{3}\}$	$\rightarrow N = 0$ class (1.3.1.3, 4)(1.3.1.1)
$1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class (1.3.1.2)(1.3.1.2)}$
$1 F_0 + 0 F_1 + \frac{2}{3} F_2 + 0 F_3 \in \{+\frac{1}{6}, -\frac{5}{6}\}$	$\rightarrow N = 0 * \text{class (1.3.1.2)(1.3.1.5, 6)}$
$1 F_0 + \frac{2}{3} F_1 + 0 F_2 + 0 F_3 \in \{+\frac{1}{6}, -\frac{5}{6}\}$	$\rightarrow N = 0 * \text{class (1.3.1.5, 6)(1.3.1.2)}$
$1 F_0 + \frac{2}{3} F_1 + \frac{2}{3} F_2 + 0 F_3 \in \{-\frac{1}{6}, +\frac{5}{6}\}$	$\rightarrow N = 0 * \text{class (1.3.1.w)(1.3.1.w)}$
	w/ $w = 5, 6$
$1 F_0 + \frac{2}{3} F_1 - \frac{2}{3} F_2 + 0 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class (1.3.1.w)(1.3.1.w')}$
	w/ $(w, w') = (5, 6), (6, 5)$

The second and the third GSOPs independently reduce the initial $N = 4$ ST-SUSY to either $N = 2$ or $N = 0$. If both are applied only $N = 0$ results.

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_5 = m \times (\text{st})S(3, 1, 1, 1)S(3, 3, 1, 1)^3$:

$1 F_0 + 1 F_1 + 1 F_2 \in \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, 1\}$	class (3.3.1.1)(1.3.1.1)
$1 F_0 + 1 F_1 + \frac{1}{3} F_2 \in \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, 1\}$	class (3.3.1.1)(1.3.1.3, 4)
$1 F_0 + 0 F_1 + 0 F_2 \in \{\pm\frac{1}{6}, \pm\frac{1}{2}, \pm\frac{5}{6}\}$	$\rightarrow N = 0 * \text{class (3.3.1.2)(1.3.1.2)}$
$1 F_0 + 0 F_1 + \frac{2}{3} F_2 \in \{\pm\frac{1}{6}, \pm\frac{1}{2}, \pm\frac{5}{6}\}$	$\rightarrow N = 0 * \text{class (3.3.1.2)(1.3.1.5, 6)}$

The first GSOP keeps two gravitinos of each chirality in \mathbf{S}_5 , creating $N = 4$ from \mathbf{S}_5 and $5\mathbf{S}_5$. The second GSOP eliminates from \mathbf{S}_5 one gravitino of each chirality, reducing ST-SUSY to $N = 2$. ($3\mathbf{S}_5$ is equivalent to \mathbf{S}_1 , from which the ever-present GSOP (1.3.1.1)(1.3.1.3) removes all gravitinos.)

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_7 = m \times (\text{st})S(3, 1, 1, 1)^2$:

$1 F_0 + 1 F_1 \in \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, 1\}$	class (1.3.1.1)(1.3.1.1)
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$$1 F_0 + 0 F_1 \in \{\pm\frac{1}{6}, \pm\frac{1}{2}, \pm\frac{5}{6}\} \rightarrow N = 0 * \text{class } (1.3.1.2)(1.3.1.2)$$

The first GSOP keeps one gravitino of each chirality in \mathbf{S}_7 , giving $N = 2$ from \mathbf{S}_7 and $5\mathbf{S}_7$. The GSOPs for $3\mathbf{S}_7$ are the same as those for \mathbf{S}_1 ; the BVs giving GSOP class (1.3.1.1)(1.3.1.1) for \mathbf{S}_7 divide up to generate the first five GSOP classes for $3\mathbf{S}_7 = \mathbf{S}_1$. Together GSOP classes (1.3.1.1)(1.3.1.1) and (1.3.1.3)(1.3.1.3) from \mathbf{V}_0 and \mathbf{S}_7 , respectively, keep two gravitinos of each chirality in $3\mathbf{S}_7$. The other possible GSOPs either keep or eliminate all of $3\mathbf{S}_7$'s gravitinos. Thus, the $\{m\mathbf{S}_7\}$ set can only produce $N = 4, 2$, or 0 ST-SUSY.

VI. LONGEST CYCLIC PERMUTATIONS: $3 \cdot 2 \cdot 1$ in $S(3, 1, 1, 1)S(2, 1, 1, 1)S(1, 1, 1, 1)$ ALLOWED GRAVITINO GENERATORS: $\mathbf{S}_1, \mathbf{S}_5$

POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(3, 3, 1, 1)S(2, 2, 1, 1)S(1, 1, 1, 1)$:

$$\begin{aligned} 1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 &\in \{0, 1\} && \text{class } (1.3.1.1)(1.2.1.1)(1.1.1.1) \\ 1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 &\in \{0, 1\} && \text{class } (1.3.1.2)(1.2.1.3)(1.1.1.1) \\ 1 F_0 + 1 F_1 + 1 F_2 + 0 F_3 &\in \{\pm\frac{1}{2}\} && \rightarrow N = 0 * \text{class } (1.3.1.1)(1.2.1.2)(1.1.1.2) \\ 1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 &\in \{\pm\frac{1}{2}\} && \rightarrow N = 0 * \text{class } (1.3.1.2)(1.2.1.4)(1.1.1.2) \\ 1 F_0 + \frac{1}{3} F_1 + 1 F_2 + 1 F_3 &\in \{+\frac{1}{3}, -\frac{2}{3}\} && \rightarrow N = 0 * \text{class } (1.3.1.3)(1.2.1.1)(1.1.1.1) \\ 1 F_0 + \frac{1}{3} F_1 + 1 F_2 + 0 F_3 &\in \{-\frac{1}{6}, +\frac{5}{6}\} && \rightarrow N = 0 * \text{class } (1.3.1.3)(1.2.1.2)(1.1.1.2) \\ 1 F_0 + \frac{2}{3} F_1 + 0 F_2 + 1 F_3 &\in \{-\frac{1}{3}, +\frac{2}{3}\} && \rightarrow N = 0 * \text{class } (1.3.1.4)(1.2.1.3)(1.1.1.1) \\ 1 F_0 + \frac{2}{3} F_1 + 0 F_2 + 0 F_3 &\in \{+\frac{1}{6}, -\frac{5}{6}\} && \rightarrow N = 0 * \text{class } (1.3.1.4)(1.2.1.4)(1.1.1.2) \end{aligned}$$

The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$.

POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(3, 3, 1, 1)S(2, 2, 1, 2)S(1, 1, 1, 1)$:

$$\begin{aligned} 1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 &\in \{0, 1\} && \text{class } (1.3.1.1)(1.2.2.1)(1.1.1.1) \\ 1 F_0 + 0 F_1 + 1 F_2 + 0 F_3 &\in \{0, 1\} && \text{class } (1.3.1.2)(1.2.2.1)(1.1.1.2) \\ 1 F_0 + 1 F_1 + 0 F_2 + 1 F_3 &\in \{\pm\frac{1}{2}\} && \rightarrow N = 0 * \text{class } (1.3.1.1)(1.2.2.2)(1.1.1.1) \\ 1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 &\in \{\pm\frac{1}{2}\} && \rightarrow N = 0 * \text{class } (1.3.1.2)(1.2.2.2)(1.1.1.2) \\ 1 F_0 + 1 F_1 + \frac{1}{2} F_2 + 1 F_3 &\in \{+\frac{1}{4}, -\frac{3}{4}\} && \rightarrow N = 0 * \text{class } (1.3.1.1)(1.2.2.3, 4)(1.1.1.1) \\ 1 F_0 + 0 F_1 + \frac{1}{2} F_2 + 0 F_3 &\in \{+\frac{1}{4}, -\frac{3}{4}\} && \rightarrow N = 0 * \text{class } (1.3.1.2)(1.2.2.3, 4)(1.1.1.2) \\ 1 F_0 + \frac{1}{3} F_1 + 1 F_2 + 1 F_3 &\in \{+\frac{1}{3}, -\frac{2}{3}\} && \rightarrow N = 0 \quad \text{class } (1.3.1.3, 4)(1.2.2.1)(1.1.1.1) \\ 1 F_0 + \frac{1}{3} F_1 + 0 F_2 + 1 F_3 &\in \{-\frac{1}{6}, +\frac{5}{6}\} && \rightarrow N = 0 * \text{class } (1.3.1.3, 4)(1.2.2.2)(1.1.1.1) \\ 1 F_0 + \frac{1}{3} F_1 + \frac{1}{2} F_2 + 1 F_3 &\in \{-\frac{5}{12}, \frac{7}{12}\} && \rightarrow N = 0 * \text{class } (1.3.1.w)(1.2.2.w)(1.1.1.1) \\ &&& \text{w/ } w = 3, 4 \\ 1 F_0 + \frac{1}{3} F_1 - \frac{1}{2} F_2 + 1 F_3 &\in \{+\frac{1}{12}, -\frac{11}{12}\} && \rightarrow N = 0 * \text{class } (1.3.1.w)(1.2.2.w')(1.1.1.1) \end{aligned}$$

$$\begin{aligned}
& w / (w, w') = (3, 4), (4, 3) \\
1 F_0 + \frac{2}{3} F_1 + 1 F_2 + 0 F_3 \in \{-\frac{1}{3}, +\frac{2}{3}\} & \rightarrow N = 0 \quad \text{class } (1.3.1.5, 6)(1.2.2.1)(1.1.1.2) \\
1 F_0 + \frac{2}{3} F_1 + 0 F_2 + 0 F_3 \in \{+\frac{1}{6}, -\frac{5}{6}\} & \rightarrow N = 0 * \text{class } (1.3.1.5, 6)(1.2.2.2)(1.1.1.2) \\
1 F_0 + \frac{2}{3} F_1 + \frac{1}{2} F_2 + 0 F_3 \in \{-\frac{1}{12}, \frac{11}{12}\} & \rightarrow N = 0 * \text{class } (1.3.1.w)(1.2.2.w')(1.1.1.2) \\
& w / (w, w') = (5, 3), (6, 4) \\
1 F_0 + \frac{2}{3} F_1 - \frac{1}{2} F_2 + 0 F_3 \in \{+\frac{5}{12}, -\frac{7}{12}\} & \rightarrow N = 0 * \text{class } (1.3.1.w)(1.2.2.w')(1.1.1.2) \\
& w / (w, w') = (5, 4), (5, 3)
\end{aligned}$$

The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$.

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_5 = m \times (\text{st})S(3, 1, 1, 1)S(2, 2, 1, 1)S(1, 1, 1, 1)$:

$$\begin{aligned}
1 F_0 + 1 F_1 + 1 F_2 \in \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, 1\} & \quad \text{class } (3.3.1.1)(1.2.1.1)(1.1.1.1) \\
1 F_0 + 0 F_1 + 1 F_2 \in \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, 1\} & \quad \text{class } (3.3.1.2)(1.2.1.3)(1.1.1.1) \\
1 F_0 + 1 F_1 + 0 F_2 \in \{\pm\frac{1}{6}, \pm\frac{1}{2}, \pm\frac{5}{6}\} & \rightarrow N = 0 * \text{class } (3.3.1.1)(1.2.1.2)(1.1.1.2) \\
1 F_0 + 0 F_1 + 0 F_2 \in \{\pm\frac{1}{6}, \pm\frac{1}{2}, \pm\frac{5}{6}\} & \rightarrow N = 0 * \text{class } (3.3.1.2)(1.2.1.4)(1.1.1.2)
\end{aligned}$$

The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$ or $N = 0$. All gravitinos are removed from $3\mathbf{S}_5$ for the reasons discussed previously.

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_5 = m \times (\text{st})S(3, 1, 1, 1)S(2, 2, 1, 2)S(1, 1, 1, 1)$:

$$\begin{aligned}
1 F_0 + 1 F_1 + 1 F_2 \in \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, 1\} & \quad \text{class } (3.3.1.1)(1.1.1.1)(1.2.2.1) \\
1 F_0 + 0 F_1 + 1 F_2 \in \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, 1\} & \quad \text{class } (3.3.1.2)(1.1.1.2)(1.2.2.1) \\
1 F_0 + 1 F_1 + 0 F_2 \in \{\pm\frac{1}{6}, \pm\frac{1}{2}, \pm\frac{5}{6}\} & \rightarrow N = 0 * \text{class } (3.3.1.1)(1.1.1.1)(1.2.2.2) \\
1 F_0 + 0 F_1 + 0 F_2 \in \{\pm\frac{1}{6}, \pm\frac{1}{2}, \pm\frac{5}{6}\} & \rightarrow N = 0 * \text{class } (3.3.1.2)(1.1.1.2)(1.2.2.2) \\
1 F_0 + 1 F_1 + \frac{1}{2} F_2 \in \{-\frac{1}{12}, \frac{3}{12}, -\frac{5}{12}, \frac{7}{12}, -\frac{9}{12}, \frac{11}{12}\} & \\
& \rightarrow N = 0 * \text{class } (3.3.1.1)(1.1.1.1)(1.2.2.3, 4) \\
1 F_0 + 0 F_1 + \frac{1}{2} F_2 \in \{-\frac{1}{12}, \frac{3}{12}, -\frac{5}{12}, \frac{7}{12}, -\frac{9}{12}, \frac{11}{12}\} & \\
& \rightarrow N = 0 * \text{class } (3.3.1.2)(1.1.1.2)(1.2.2.3, 4)
\end{aligned}$$

The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$ or $N = 0$.

V. LONGEST CYCLIC PERMUTATIONS: $3 \cdot 1 \cdot 1 \cdot 1$ in $S(3, 1, 1, 1)S(1, 1, 1, 1)^3$

ALLOWED GRAVITINO GENERATORS: $\mathbf{S}_1, \mathbf{S}_5$

.....
POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(3, 3, 1, 1)S(1, 1, 1, 1)^3$:

$$\begin{aligned}
1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\} & \quad \text{class } (1.3.1.1)(1.1.1.1)^3 \\
1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 \in \{0, 1\} & \quad \text{class } (1.3.1.2)(1.1.1.2)(1.1.1.1)^2
\end{aligned}$$

$$\begin{aligned}
1 F_0 + 1 F_1 + 1 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.3.1.1)(1.1.1.1)(1.1.1.2)^2 \\
1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.3.1.2)(1.1.1.2)^3 \\
1 F_0 + \frac{1}{3} F_1 + 1 F_2 + 1 F_3 \in \{+\frac{1}{3}, -\frac{2}{3}\} &\rightarrow N = 0 * \text{class } (1.3.1.3)(1.1.1.1)^3 \\
1 F_0 + \frac{1}{3} F_1 + 1 F_2 + 0 F_3 \in \{-\frac{1}{6}, +\frac{5}{6}\} &\rightarrow N = 0 * \text{class } (1.3.1.3)(1.1.1.1)(1.1.1.2)^2 \\
1 F_0 + \frac{2}{3} F_1 + 0 F_2 + 1 F_3 \in \{-\frac{1}{3}, +\frac{2}{3}\} &\rightarrow N = 0 * \text{class } (1.3.1.4)(1.1.1.2)(1.1.1.1)^2 \\
1 F_0 + \frac{2}{3} F_1 + 0 F_2 + 0 F_3 \in \{+\frac{1}{6}, -\frac{5}{6}\} &\rightarrow N = 0 * \text{class } (1.3.1.4)(1.1.1.2)^3
\end{aligned}$$

The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$.

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 POSSIBLE GSO PROJECTIONS ON $m\mathcal{S}_5 = m \times (\text{st})S(3, 1, 1, 1)S(1, 1, 1, 1)^3$:

$$\begin{aligned}
 1 F_0 +1 F_1 +1 F_2 &\in \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, 1\} && \text{class } (3.3.1.1)(1.1.1.1)^3 \\
 1 F_0 +0 F_1 +1 F_2 &\in \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, 1\} && \text{class } (3.3.1.2)(1.1.1.2)(1.1.1.1)^2 \\
 1 F_0 +1 F_1 +0 F_2 &\in \{\pm\frac{1}{6}, \pm\frac{1}{2}, \pm\frac{5}{6}\} &\rightarrow N = 0 * \text{class } (3.3.1.1)(1.1.1.1)(1.1.1.1)^2 \\
 1 F_0 +0 F_1 +0 F_2 &\in \{\pm\frac{1}{6}, \pm\frac{1}{2}, \pm\frac{5}{6}\} &\rightarrow N = 0 * \text{class } (3.3.1.2)(1.1.1.2)^3
 \end{aligned}$$

The second GSOP reduces the initial $N = 4$ ST-SUSY to $N = 2$ or $N = 0$.

IV. LONGEST CYCLIC PERMUTATIONS: $2 \cdot 2 \cdot 2$ in $S(2, 1, 1, 1)^3$

ALLOWED GRAVITINO GENERATORS: $\mathcal{S}_1, \mathcal{S}_3$

.....
 POSSIBLE GSO PROJECTIONS ON $\mathcal{S}_1 = (\text{st})S(2, 2, 1, 1)^2S(2, 2, 1, 2)$:

$$\begin{aligned}
 1 F_0 +1 F_1 +1 F_2 +1 F_3 &\in \{0, 1\} && \text{class } (1.2.1.1)^2(1.2.2.1) \\
 1 F_0 +1 F_1 +0 F_2 +0 F_3 &\in \{0, 1\} && \text{class } (1.2.1.2)^2(1.2.2.2) \\
 1 F_0 +0 F_1 +1 F_2 +0 F_3 &\in \{0, 1\} && \text{class } (1.2.1.3)^2(1.2.2.2) \\
 1 F_0 +0 F_1 +0 F_2 +1 F_3 &\in \{0, 1\} && \text{class } (1.2.1.4)^2(1.2.2.1) \\
 1 F_0 +1 F_1 +1 F_2 +0 F_3 &\in \{\pm\frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.1.1)^2(1.2.2.2) \\
 1 F_0 +1 F_1 +0 F_2 +1 F_3 &\in \{\pm\frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.1.2)^2(1.2.2.1) \\
 1 F_0 +0 F_1 +1 F_2 +1 F_3 &\in \{\pm\frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.1.3)^2(1.2.2.1) \\
 1 F_0 +0 F_1 +0 F_2 +0 F_3 &\in \{\pm\frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.1.4)^2(1.2.2.2) \\
 1 F_0 +1 F_1 +1 F_2 +\frac{1}{2} F_3 &\in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.1.1)^2(1.2.2.3, 4) \\
 1 F_0 +1 F_1 +0 F_2 +\frac{1}{2} F_3 &\in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.1.2)^2(1.2.2.3, 4) \\
 1 F_0 +0 F_1 +1 F_2 +\frac{1}{2} F_3 &\in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.1.3)^2(1.2.2.3, 4) \\
 1 F_0 +0 F_1 +0 F_2 +\frac{1}{2} F_3 &\in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.1.4)^2(1.2.2.3, 4)
 \end{aligned}$$

The first four GSOPs give the standard NAHE set of GSOPs, although the basis vectors generating these GSOPs do not correspond to the NAHE basis vectors. The combination of any three of these four GSOPs produce $N = 1$ ST-SUSY, independent of their RHS values. The remaining GSOP either keeps or breaks $N = 1$, depending upon its RHS value. However, we cannot count this as a category IV solution. It is in fact a category III solution since the first four GSOP classes do not come from states with $2 \cdot 2 \cdot 2$ cycles, but from $2 \cdot 2 \cdot 1 \cdot 1$ and $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ cycles.

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 POSSIBLE GSO PROJECTIONS ON $\mathcal{S}_1 = (\text{st})S(2, 2, 1, 2)^3$:

GROUP 1: GSOPs generated from an embedded $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ permutation

$1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\}$	class $(1.2.2.1)^3$
$1 F_0 + 1 F_1 + 0 F_2 + 0 F_3 \in \{0, 1\}$	class $(1.2.2.1)(1.2.2.2)^2$
$1 F_0 + 0 F_1 + 1 F_2 + 0 F_3 \in \{0, 1\}$	class $(1.2.2.2)(1.2.2.1)(1.2.2.2)$
$1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 \in \{0, 1\}$	class $(1.2.2.2)^2(1.2.2.1)$

GROUP 2: GSOPs generated from an embedded $2 \cdot 2 \cdot 1 \cdot 1$ permutation

$1 F_0 + \frac{1}{2} F_1 + \frac{1}{2} F_2 + 1 F_3 \in \{\pm \frac{1}{2}\}$	class $(1.2.2.w)^2(1.2.2.1)$ w/ $w = 3, 4$
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + 1 F_3 \in \{0, 1\}$	class $(1.2.2.w)(1.2.2.w')(1.2.2.1)$ w/ $(w, w') = (3, 4), (4, 3)$
$1 F_0 + \frac{1}{2} F_1 + \frac{1}{2} F_2 + 0 F_3 \in \{0, 1\}$	class $(1.2.2.w)^2(1.2.2.2)$ w/ $w = 3, 4$
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + 0 F_3 \in \{\pm \frac{1}{2}\}$	class $(1.2.2.w)(1.2.2.w')(1.2.2.2)$ w/ $(w, w') = (3, 4), (4, 3)$

GROUP 3: GSOPs generated from an embedded $2 \cdot 1 \cdot 1 \cdot 2$ permutation

$1 F_0 + \frac{1}{2} F_1 + 1 F_2 + \frac{1}{2} F_3 \in \{\pm \frac{1}{2}\}$	class $(1.2.2.w)(1.2.2.1)(1.2.2.w)$ w/ $w = 3, 4$
$1 F_0 + \frac{1}{2} F_1 + 1 F_2 - \frac{1}{2} F_3 \in \{0, 1\}$	class $(1.2.2.w)(1.2.2.1)(1.2.2.w')$ w/ $(w, w') = (3, 4), (4, 3)$
$1 F_0 + \frac{1}{2} F_1 + 0 F_2 + \frac{1}{2} F_3 \in \{0, 1\}$	class $(1.2.2.w)(1.2.2.2)(1.2.2.w)$ w/ $w = 3, 4$
$1 F_0 + \frac{1}{2} F_1 + 0 F_2 - \frac{1}{2} F_3 \in \{\pm \frac{1}{2}\}$	class $(1.2.2.w)(1.2.2.2)(1.2.2.w')$ w/ $(w, w') = (3, 4), (4, 3)$

GROUP 4: GSOPs generated from an embedded $1 \cdot 1 \cdot 2 \cdot 2$ permutation

$1 F_0 + 1 F_1 + \frac{1}{2} F_2 + \frac{1}{2} F_3 \in \{\pm \frac{1}{2}\}$	class $(1.2.2.1)(1.2.2.w)^2$ w/ $w = 3, 4$
$1 F_0 + 1 F_1 + \frac{1}{2} F_2 - \frac{1}{2} F_3 \in \{0, 1\}$	class $(1.2.2.1)(1.2.2.w)(1.2.2.w')$ w/ $(w, w') = (3, 4), (4, 3)$
$1 F_0 + 0 F_1 + \frac{1}{2} F_2 + \frac{1}{2} F_3 \in \{0, 1\}$	class $(1.2.2.2)(1.2.2.w)^2$ w/ $w = 3, 4$
$1 F_0 + 0 F_1 + \frac{1}{2} F_2 - \frac{1}{2} F_3 \in \{\pm \frac{1}{2}\}$	class $(1.2.2.2)(1.2.2.w)(1.2.2.w')$ w/ $(w, w') = (3, 4), (4, 3)$
$1 F_0 + 1 F_1 + 1 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0 * \text{class } (1.2.2.1)^2(1.2.2.2)$

$$\begin{aligned}
1 F_0 + 1 F_1 + 0 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.2.1)(1.2.2.2)(1.2.2.1) \\
1 F_0 + 0 F_1 + 1 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.2.2)(1.2.2.1)^2 \\
1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.2.2)^3 \\
1 F_0 + 1 F_1 + 1 F_2 + \frac{1}{2} F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.1)^2(1.2.2.3, 4) \\
1 F_0 + 1 F_1 + 0 F_2 + \frac{1}{2} F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.1)(1.2.2.2)(1.2.2.3, 4) \\
1 F_0 + 0 F_1 + 1 F_2 + \frac{1}{2} F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.2)(1.2.2.1)(1.2.2.3, 4) \\
1 F_0 + 0 F_1 + 0 F_2 + \frac{1}{2} F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.2)^2(1.2.2.3, 4) \\
1 F_0 + 1 F_1 + \frac{1}{2} F_2 + 1 F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.1)(1.2.2.3, 4)(1.2.2.1) \\
1 F_0 + 1 F_1 + \frac{1}{2} F_2 + 0 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.1)(1.2.2.3, 4)(1.2.2.2) \\
1 F_0 + 0 F_1 + \frac{1}{2} F_2 + 1 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.2)(1.2.2.3, 4)(1.2.2.1) \\
1 F_0 + 0 F_1 + \frac{1}{2} F_2 + 0 F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.2)(1.2.2.3, 4)(1.2.2.2) \\
1 F_0 + \frac{1}{2} F_1 + 1 F_2 + 1 F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.2.2.1)^2 \\
1 F_0 + \frac{1}{2} F_1 + 1 F_2 + 0 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.2.2.1)(1.2.2.2) \\
1 F_0 + \frac{1}{2} F_1 + 0 F_2 + 1 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.2.2.2)(1.2.2.1) \\
1 F_0 + \frac{1}{2} F_1 + 0 F_2 + 0 F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.2.2.2)^2 \\
1 F_0 + \frac{1}{2} F_1 + \frac{1}{2} F_2 + \frac{1}{2} F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.w)^3 \\
&\quad \text{w/ } w = 3, 4 \\
1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 - \frac{1}{2} F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.w)(1.2.2.w')^2 \\
&\quad \text{w/ } (w, w') = (3, 4), (4, 3) \\
1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + \frac{1}{2} F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.w)(1.2.2.w')(1.2.2.w) \\
&\quad \text{w/ } (w, w') = (3, 4), (4, 3) \\
1 F_0 + \frac{1}{2} F_1 + \frac{1}{2} F_2 - \frac{1}{2} F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.w)(1.2.2.w)(1.2.2.w') \\
&\quad \text{w/ } (w, w') = (3, 4), (4, 3)
\end{aligned}$$

Within the first sixteen GSOPs are four isomorphic maximal sets of GSOPs that can render $N = 1$ ST-SUSY. They correspond to various combinations of GSOPs from the three groups corresponding to embeddings of $1 \cdot 1 \cdot 2 \cdot 2$, $2 \cdot 1 \cdot 1 \cdot 2$, and $1 \cdot 1 \cdot 2 \cdot 2$. The first and the fourth GSOPs in one embedding can appear in the same GSOP set, as may the second and the third. An $N = 1$ set can have either a first/fourth GSOP or a second/third GSOP contribution from each of these three embedding groups, as long as the number of groups contributing second/third GSOPs is odd. The first or second GSOPs in these three groups give $N = 2$ ST-SUSY, whereas the third or fourth directly give $N = 1$. Whenever two “first or second” GSOPs or a single “third or fourth” GSOP are chosen, the RHSs must be fixed for all other GSOPs present if $N = 1$ ST-SUSY is to remain. Each maximal set contains the entire first group, which forms the actual NAHE projection of category I. The NAHE set BVs are contained in the classes of BVs generating this first group.

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 POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_3 = m \times (\text{st})S(2, 1, 1, 1)^2 S(2, 2, 1, 1)$:

$1 F_0 + 1 F_1 + 1 F_2 \in \{0, 1\}$	class $(2.2.1.1)^2(1.2.1.1)$
$1 F_0 + 1 F_1 + 0 F_2 \in \{0, 1\}$	class $(2.2.1.w)(2.2.1.w')(1.2.1.3)$ w/ $(w, w') = (2, 3), (1, 4)$
$1 F_0 + 0 F_1 + 1 F_2 \in \{0, 1\}$	class $(2.2.1.w)(2.2.1.w')(1.2.1.2)$ w/ $(w, w') = (3, 2), (4, 1)$
$1 F_0 + 0 F_1 + 0 F_2 \in \{0, 1\}$	class $(2.2.1.3)^2(1.2.1.4)$
$1 F_0 + 1 F_1 + 1 F_2 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0 * \text{class } (2.2.1.2)^2(1.2.1.1)$
$1 F_0 + 1 F_1 + 1 F_2 \in \{+\frac{1}{4}, -\frac{3}{4}\}$	$\rightarrow N = 0 * \text{class } (2.2.1.w)(2.2.1.w')(1.2.1.1)$ w/ $(w, w') = (1, 2), (2, 1)$
$1 F_0 + 1 F_1 + 0 F_2 \in \{+\frac{1}{4}, -\frac{3}{4}\}$	$\rightarrow N = 0 * \text{class } (2.2.1.w)(2.2.1.w')(1.2.1.3)$ w/ $(w, w') = (1, 3), (2, 4)$
$1 F_0 + 0 F_1 + 1 F_2 \in \{+\frac{1}{4}, -\frac{3}{4}\}$	$\rightarrow N = 0 * \text{class } (2.2.1.w)(2.2.1.w')(1.2.1.2)$ w/ $(w, w') = (3, 1), (4, 2)$
$1 F_0 + 0 F_1 + 0 F_2 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0 \quad \text{class } (2.2.1.4)^2(1.2.1.4)$
$1 F_0 + 0 F_1 + 0 F_2 \in \{+\frac{1}{4}, -\frac{3}{4}\}$	$\rightarrow N = 0 * \text{class } (2.2.1.w)(2.2.1.w')(1.2.1.4)$ w/ $(w, w') = (3, 4), (4, 3)$

The first four GSOPs imitate the NAHE set. The combination of any three of these four GSOPs produce $N = 1$ ST-SUSY independent of their RHS values. The remaining GSOP either keeps or breaks $N = 1$, depending upon its RHS value.

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 POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_3 = m \times (\text{st})S(2, 1, 1, 1)^2 S(2, 2, 1, 2)$:

$1 F_0 + 1 F_1 + 1 F_2 \in \{0, 1\}$	class $(2.2.1.1)^2(1.2.2.1)$
$1 F_0 + 1 F_1 + 0 F_2 \in \{0, 1\}$	class $(2.2.1.1)^2(1.2.2.2)$
$1 F_0 + 0 F_1 + 1 F_2 \in \{0, 1\}$	class $(2.2.1.3)^2(1.2.2.1)$
$1 F_0 + 0 F_1 + 0 F_2 \in \{0, 1\}$	class $(2.2.1.4)^2(1.2.2.2)$
$1 F_0 + 1 F_1 + 1 F_2 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0 \quad \text{class } (2.2.1.2)^2(1.2.2.1)$
$1 F_0 + 1 F_1 + 0 F_2 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0 \quad \text{class } (2.2.1.2)^2(1.2.2.2)$
$1 F_0 + 0 F_1 + 1 F_2 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0 \quad \text{class } (2.2.1.4)^2(1.2.2.1)$
$1 F_0 + 0 F_1 + 0 F_2 \in \{\pm \frac{1}{2}\}$	$\rightarrow N = 0 \quad \text{class } (2.2.1.3)^2(1.2.2.2)$
$1 F_0 + 1 F_1 + \frac{1}{2} F_2 \in \{+\frac{1}{4}, -\frac{3}{4}\}$	$\rightarrow N = 0 * \text{class } (2.2.1.1, 2)^2(1.2.2.3, 4)$
$1 F_0 + 0 F_1 + \frac{1}{2} F_2 \in \{+\frac{1}{4}, -\frac{3}{4}\}$	$\rightarrow N = 0 * \text{class } (2.2.1.3, 4)^2(1.2.2.3, 4)$

The first four GSOPs imitate the NAHE set. The combination of any three of these four GSOPs produce $N = 1$ ST-SUSY independent of their RHS values. The remaining

GSOP either keeps or breaks $N = 1$, depending upon its RHS value.

III. LONGEST CYCLIC PERMUTATIONS: $2 \cdot 2 \cdot 1 \cdot 1$ in $S(2, 1, 1, 1)^2 S(1, 1, 1, 1)^2$

ALLOWED GRAVITINO GENERATORS: $\mathbf{S}_1, \mathbf{S}_3$

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POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(2, 2, 1, 1)^2 S(1, 1, 1, 1)^2$:

$1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\}$	class $(1.2.1.1)^2(1.1.1.1)^2$
$1 F_0 + 1 F_1 + 0 F_2 + 0 F_3 \in \{0, 1\}$	class $(1.2.1.2)^2(1.1.1.2)^2$
$1 F_0 + 0 F_1 + 1 F_2 + 0 F_3 \in \{0, 1\}$	class $(1.2.1.3)^2(1.1.1.2)^2$
$1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 \in \{0, 1\}$	class $(1.2.1.4)^2(1.1.1.1)^2$
$1 F_0 + 1 F_1 + 1 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} \rightarrow N = 0$	$* \text{class } (1.2.1.1)^2(1.1.1.2)^2$
$1 F_0 + 1 F_1 + 0 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} \rightarrow N = 0$	$* \text{class } (1.2.1.2)^2(1.1.1.1)^2$
$1 F_0 + 0 F_1 + 1 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} \rightarrow N = 0$	$* \text{class } (1.2.1.3)^2(1.1.1.1)^2$
$1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} \rightarrow N = 0$	$* \text{class } (1.2.1.4)^2(1.1.1.2)^2$

These GSOPs are a subset of those for $\mathbf{S}_1 = (\text{st})S(2, 2, 1, 1)^2 S(2, 2, 1, 2)$ in IV. We replace $(1.2.2.w)$ with $(1.1.1.w)^2$ for $w = 1, 2$ and remove the GSOPs corresponding to the $(1.2.2.3, 4)$ class. The first four GSOPs give the standard NAHE set of GSOPs, although the basis vectors generating these GSOPs do not correspond to the NAHE basis vectors. The combination of any three of these four GSOPs produce $N = 1$ ST-SUSY, independent of their RHS values. The remaining GSOP either keeps or breaks $N = 1$, depending upon its RHS value.

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POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(2, 2, 1, 2)^2 S(1, 1, 1, 1)^2$:

GROUP 1: GSOPs generated from an embedded $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ permutation

$1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\}$	class $(1.2.2.1)^2(1.1.1.1)^2$
$1 F_0 + 1 F_1 + 0 F_2 + 0 F_3 \in \{0, 1\}$	class $(1.2.2.1)(1.2.2.2)(1.1.1.2)^2$
$1 F_0 + 0 F_1 + 1 F_2 + 0 F_3 \in \{0, 1\}$	class $(1.2.2.2)(1.2.2.1)(1.1.1.2)^2$
$1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 \in \{0, 1\}$	class $(1.2.2.2)^2(1.1.1.1)^2$

GROUP 2.

$1 F_0 + \frac{1}{2} F_1 + \frac{1}{2} F_2 + 1 F_3 \in \{\pm \frac{1}{2}\}$	class $(1.2.2.w)^2(1.1.1.1)^2$ w/ $w = 3, 4$
$1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + 1 F_3 \in \{0, 1\}$	class $(1.2.2.w)(1.2.2.w')(1.1.1.1)^2$ w/ $(w, w') = (3, 4), (4, 3)$
$1 F_0 + \frac{1}{2} F_1 + \frac{1}{2} F_2 + 0 F_3 \in \{0, 1\}$	class $(1.2.2.w)^2(1.1.1.2)^2$

$$\begin{array}{ll}
1 F_0 + \frac{1}{2} F_1 - \frac{1}{2} F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} & \begin{array}{l} w/ \ w = 3, 4 \\ \text{class } (1.2.2.w)(1.2.2.w')(1.1.1.2)^2 \\ w/ \ (w, w') = (3, 4), (4, 3) \end{array} \\
\\
1 F_0 + 1 F_1 + 1 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 * \text{class } (1.2.2.1)^2(1.1.1.2)^2 \\
1 F_0 + 1 F_1 + 0 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 * \text{class } (1.2.2.1)(1.2.2.2)(1.1.1.1)^2 \\
1 F_0 + 0 F_1 + 1 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 * \text{class } (1.2.2.2)(1.2.2.1)(1.1.1.1)^2 \\
1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 * \text{class } (1.2.2.2)^2(1.1.1.2)^2 \\
\\
1 F_0 + 1 F_1 + \frac{1}{2} F_2 + 1 F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} & \rightarrow N = 0 * \text{class } (1.2.2.1)(1.2.2.3, 4)(1.1.1.1)^2 \\
1 F_0 + 1 F_1 + \frac{1}{2} F_2 + 0 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} & \rightarrow N = 0 * \text{class } (1.2.2.1)(1.2.2.3, 4)(1.1.1.2)^2 \\
1 F_0 + 0 F_1 + \frac{1}{2} F_2 + 1 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} & \rightarrow N = 0 * \text{class } (1.2.2.2)(1.2.2.3, 4)(1.1.1.1)^2 \\
1 F_0 + 0 F_1 + \frac{1}{2} F_2 + 0 F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} & \rightarrow N = 0 * \text{class } (1.2.2.2)(1.2.2.3, 4)(1.1.1.2)^2 \\
1 F_0 + \frac{1}{2} F_1 + 1 F_2 + 1 F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} & \rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.2.2.1)(1.1.1.1)^2 \\
1 F_0 + \frac{1}{2} F_1 + 1 F_2 + 0 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} & \rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.2.2.1)(1.1.1.2)^2 \\
1 F_0 + \frac{1}{2} F_1 + 0 F_2 + 1 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} & \rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.2.2.2)(1.1.1.1)^2 \\
1 F_0 + \frac{1}{2} F_1 + 0 F_2 + 0 F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} & \rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.2.2.2)(1.1.1.2)^2
\end{array}$$

These GSOPs are a subset of those for $\mathcal{S}_1 = (\text{st})S(2, 2, 1, 1)^2S(2, 2, 1, 2)$ in IV. Of the four groups only the first two survive. We replace $(1.2.2.w)$ with $(1.1.1.w)^2$ for $w = 1, 2$ and remove the GSOPs corresponding to the $(1.2.2.3, 4)$ class. Only two maximal sets yielding $N = 1$ ST-SUSY remain. Either the first and fourth GSOPs in the second group or the second and third GSOPs can be combined with the first group. The NAHE set BVs are contained in the classes of BVs generating the first four GSOPs.

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POSSIBLE GSO PROJECTIONS ON $m\mathcal{S}_3 = m \times (\text{st})S(2, 1, 1, 1)^2S(1, 1, 1, 1)^2$:

$$\begin{array}{ll}
1 F_0 + 1 F_1 + 1 F_2 \in \{0, 1\} & \text{class } (2.2.1.1)^2(1.1.1.1)^2 \\
1 F_0 + 1 F_1 + 0 F_2 \in \{0, 1\} & \begin{array}{l} \text{class } (2.2.1.w)(2.2.1.w')(1.1.1.2)(1.1.1.1) \\ w/ \ (w, w') = (2, 3), (1, 4) \end{array} \\
1 F_0 + 0 F_1 + 1 F_2 \in \{0, 1\} & \begin{array}{l} \text{class } (2.2.1.w)(2.2.1.w')(1.1.1.1)(1.1.1.2) \\ w/ \ (w, w') = (3, 2), (4, 1) \end{array} \\
1 F_0 + 0 F_1 + 0 F_2 \in \{0, 1\} & \text{class } (2.2.1.3)^2(1.1.1.2)^2 \\
1 F_0 + 1 F_1 + 1 F_2 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 * \text{class } (2.2.1.2)^2(1.1.1.1)^2 \\
1 F_0 + 1 F_1 + 1 F_2 \in \{+\frac{1}{4}, -\frac{3}{4}\} & \begin{array}{l} \rightarrow N = 0 * \text{class } (2.2.1.w)(2.2.1.w')(1.1.1.1)^2 \\ w/ \ (w, w') = (1, 2), (2, 1) \end{array} \\
1 F_0 + 1 F_1 + 0 F_2 \in \{+\frac{1}{4}, -\frac{3}{4}\} & \begin{array}{l} \rightarrow N = 0 * \text{class } (2.2.1.w)(2.2.1.w')(1.1.1.2)(1.1.1.1) \\ w/ \ (w, w') = (1, 3), (2, 4) \end{array} \\
1 F_0 + 0 F_1 + 1 F_2 \in \{+\frac{1}{4}, -\frac{3}{4}\} & \rightarrow N = 0 * \text{class } (2.2.1.w)(2.2.1.w')(1.1.1.1)(1.1.1.2)
\end{array}$$

$$\begin{aligned}
& w / (w, w') = (3, 1), (4, 2) \\
1 F_0 + 0 F_1 + 0 F_2 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 \quad \text{class } (2.2.1.4)^2(1.1.1.2)^2 \\
1 F_0 + 0 F_1 + 0 F_2 \in \{+\frac{1}{4}, -\frac{3}{4}\} & \rightarrow N = 0 * \text{class } (2.2.1.w)(2.2.1.w')(1.1.1.2)^2 \\
& w / (w, w') = (3, 4), (4, 3)
\end{aligned}$$

These GSOPs form the same set as those for $\mathbf{S}_3 = (\text{st})S(2, 1, 1, 1)^2S(2, 2, 1, 1)$ in IV. We replace (1.2.1.1) with $(1.1.1.1)^2$, (1.2.1.2) with $(1.1.1.1)(1.1.1.2)$, (1.2.1.3) with $(1.1.1.2)(1.1.1.1)$, and (1.2.1.4) with $(1.1.1.2)^2$. The first four GSOPs imitate the NAHE set. The combination of any three of these four GSOPs produce $N = 1$ ST-SUSY independent of their RHS values. The remaining GSOP either keeps or breaks $N = 1$, depending upon its RHS value.

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POSSIBLE GSO PROJECTIONS ON $m\mathbf{S}_3 = m \times (\text{st})S(2, 1, 1, 1)^2S(1, 1, 1, 1)^2$:

$$\begin{aligned}
1 F_0 + 1 F_1 + 1 F_2 \in \{0, 1\} & \quad \text{class } (2.2.1.1)^2(1.1.1.1)^2 \\
1 F_0 + 1 F_1 + 0 F_2 \in \{0, 1\} & \quad \text{class } (2.2.1.1)^2(1.1.1.2)^2 \\
1 F_0 + 0 F_1 + 1 F_2 \in \{0, 1\} & \quad \text{class } (2.2.1.3)^2(1.1.1.1)^2 \\
1 F_0 + 0 F_1 + 0 F_2 \in \{0, 1\} & \quad \text{class } (2.2.1.4)^2(1.1.1.2)^2 \\
1 F_0 + 1 F_1 + 1 F_2 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 \quad \text{class } (2.2.1.2)^2(1.1.1.1)^2 \\
1 F_0 + 1 F_1 + 0 F_2 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 \quad \text{class } (2.2.1.2)^2(1.1.1.2)^2 \\
1 F_0 + 0 F_1 + 1 F_2 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 \quad \text{class } (2.2.1.4)^2(1.1.1.1)^2 \\
1 F_0 + 0 F_1 + 0 F_2 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 \quad \text{class } (2.2.1.3)^2(1.1.1.2)^2
\end{aligned}$$

These GSOPs are a subset of those for $\mathbf{S}_3 = (\text{st})S(2, 2, 1, 1)^2S(2, 2, 1, 2)$ in IV. We replace (1.2.2.w) with $(1.1.1.w)^2$ for $w = 1, 2$ and remove of GSOPs corresponding to the (1.2.2.3, 4) class. The first four GSOPs imitate the NAHE set. The combination of any three of these four GSOPs produce $N = 1$ ST-SUSY independent of their RHS values. The remaining GSOP either keeps or breaks $N = 1$, depending upon its RHS value.

II. LONGEST CYCLIC PERMUTATIONS: $2 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ in $S(2, 1, 1, 1)S(1, 1, 1, 1)^4$

ALLOWED GRAVITINO GENERATORS: \mathbf{S}_1

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POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(2, 2, 1, 1)S(1, 1, 1, 1)^4$:

$$\begin{aligned}
1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\} & \quad \text{class } (1.2.1.1)(1.1.1.1)^4 \\
1 F_0 + 1 F_1 + 0 F_2 + 0 F_3 \in \{0, 1\} & \quad \text{class } (1.2.1.2)(1.1.1.1)(1.1.1.2)^3 \\
1 F_0 + 0 F_1 + 1 F_2 + 0 F_3 \in \{0, 1\} & \quad \text{class } (1.2.1.3)(1.1.1.2)(1.1.1.1)(1.1.1.2)^2 \\
1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 \in \{0, 1\} & \quad \text{class } (1.2.1.4)(1.1.1.2)^2(1.1.1.1)^2 \\
1 F_0 + 1 F_1 + 1 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} & \rightarrow N = 0 * \text{class } (1.2.1.1)(1.1.1.1)^2(1.1.1.2)^2
\end{aligned}$$

$$\begin{aligned}
1 F_0 + 1 F_1 + 0 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.1.2)(1.1.1.1)(1.1.1.2)(1.1.1.1)^2 \\
1 F_0 + 0 F_1 + 1 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.1.3)(1.1.1.2)(1.1.1.1)^3 \\
1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.1.4)(1.1.1.2)^4
\end{aligned}$$

These GSOPs form the same set as those for $\mathbf{S}_1 = (\text{st})S(2, 2, 1, 1)^2 S(1, 1, 1, 1)$ in III. We replace the second (1.2.1.1) with $(1.1.1.1)^2$, (1.2.1.2) with $(1.1.1.1)(1.1.1.2)$, (1.2.1.3) with $(1.1.1.2)(1.1.1.1)$, and (1.2.1.4) with $(1.1.1.2)^2$. The first four GSOPs give the standard NAHE set of GSOPs, although the basis vectors generating these GSOPs do not correspond to the NAHE basis vectors. The combination of any three of these four GSOPs produce $N = 1$ ST-SUSY, independent of their RHS values. The remaining GSOP either keeps or breaks $N = 1$, depending upon its RHS value.

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POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(2, 2, 1, 2)S(1, 1, 1, 1)^4$:

$$\begin{aligned}
1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\} &\text{class } (1.2.2.1)(1.1.1.1)^4 \\
1 F_0 + 1 F_1 + 0 F_2 + 0 F_3 \in \{0, 1\} &\text{class } (1.2.2.1)(1.1.1.2)^4 \\
1 F_0 + 0 F_1 + 1 F_2 + 0 F_3 \in \{0, 1\} &\text{class } (1.2.2.2)(1.1.1.1)^2(1.1.1.2)^2 \\
1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 \in \{0, 1\} &\text{class } (1.2.2.2)(1.1.1.2)^2(1.1.1.1)^2 \\
1 F_0 + 1 F_1 + 1 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.2.1)(1.1.1.1)^2(1.1.1.2)^2 \\
1 F_0 + 1 F_1 + 0 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.2.1)(1.1.1.2)^2(1.1.1.1)^2 \\
1 F_0 + 0 F_1 + 1 F_2 + 1 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.2.2)(1.1.1.1)^4 \\
1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm \frac{1}{2}\} &\rightarrow N = 0 * \text{class } (1.2.2.2)(1.1.1.1)^4 \\
1 F_0 + \frac{1}{2} F_1 + 1 F_2 + 1 F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.1.1.1)^4 \\
1 F_0 + \frac{1}{2} F_1 + 1 F_2 + 0 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.1.1.1)^2(1.1.1.2)^2 \\
1 F_0 + \frac{1}{2} F_1 + 0 F_2 + 1 F_3 \in \{-\frac{1}{4}, +\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.1.1.2)^2(1.1.1.1)^2 \\
1 F_0 + \frac{1}{2} F_1 + 0 F_2 + 0 F_3 \in \{+\frac{1}{4}, -\frac{3}{4}\} &\rightarrow N = 0 * \text{class } (1.2.2.3, 4)(1.1.1.2)^4
\end{aligned}$$

These GSOPs are a subset of those for $\mathbf{S}_1 = (\text{st})S(2, 2, 1, 2)^2 S(1.1.1.1)^2$ in III. We replace the second (1.2.2.w) with $(1.1.1.w)^2$ for $w = 1, 2$ and remove the GSOPs corresponding to the (1.2.2.3, 4) class. However, the first four GSOPs are equivalent to the $N = 1$ solution for $\mathbf{S}_1 = (\text{st})S(1.1.1.1)^6$ in I. The NAHE set BVs are contained in the classes of BVs generating these four GSOPs.

I. LONGEST CYCLIC PERMUTATIONS: $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$ in $S(1, 1, 1, 1)^6$

ALLOWED GRAVITINO GENERATORS: \mathbf{S}_1

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POSSIBLE GSO PROJECTIONS ON $\mathbf{S}_1 = (\text{st})S(1, 1, 1, 1)^6$:

$$1 F_0 + 1 F_1 + 1 F_2 + 1 F_3 \in \{0, 1\} \quad \text{class } (1.1.1.1)^6$$

$1 F_0 + 1 F_1 + 0 F_2 + 0 F_3 \in \{0, 1\}$	class $(1.1.1.1)^2(1.1.1.2)^4$
$1 F_0 + 0 F_1 + 1 F_2 + 0 F_3 \in \{0, 1\}$	class $(1.1.1.1)^2(1.1.1.2)^2(1.1.1.1)^2$
$1 F_0 + 0 F_1 + 0 F_2 + 1 F_3 \in \{0, 1\}$	class $(1.1.1.2)^4(1.1.1.1)^2$
$1 F_0 + 1 F_1 + 1 F_2 + 0 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class } (1.1.1.1)^4(1.1.1.1)^2$
$1 F_0 + 1 F_1 + 0 F_2 + 1 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class } (1.1.1.1)^2(1.1.1.2)^2(1.1.1.1)^2$
$1 F_0 + 0 F_1 + 1 F_2 + 1 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class } (1.1.1.2)^2(1.1.1.1)^4$
$1 F_0 + 0 F_1 + 0 F_2 + 0 F_3 \in \{\pm\frac{1}{2}\}$	$\rightarrow N = 0 * \text{class } (1.1.1.2)^6$

The first four GSOPs form the NAHE solution set for $N = 1$ ST-SUSY. This is the set used in all string models to date. The NAHE set BVs are contained in the classes of BVs generating these four GSOPs. The combination of any three of these four GSOPs produce $N = 1$ ST-SUSY, independent of their RHS values. The remaining GSOP either keeps or breaks $N = 1$, depending upon its RHS value,

This finishes our classification of allowed sets of GSOPs acting on a given source of gravitinos in heterotic free fermionic strings. The findings of our exhaustive search can be summarized as follows:

1. If sectors involving non-disjoint cycles of length 3, 5, or 6 (corresponding to \mathbb{Z}_6 , \mathbb{Z}_{10} , and \mathbb{Z}_{12} projections, respectively) are present in a model, then $N = 1$ ST-SUSY is forbidden. Thus, \mathbf{S}_5 , \mathbf{S}_7 , and \mathbf{S}_{10} as gravitino generators can only result in $N = 4, 2$, or 0 ST-SUSY.
2. $N = 1$ ST-SUSY is possible for \mathbf{S}_1 , \mathbf{S}_3 , and \mathbf{S}_9 . There are six general categories of reduction set solutions (IX, VIII, IV, III, II and I) for \mathbf{S}_1 , three (IX, IV, III) for \mathbf{S}_3 , and one (IX) for \mathbf{S}_9 .

Let us define the effective order of a GSOP for a specific gravitino generator \mathbf{S}_i as the order of the components of the BV giving GSOP's \mathbf{F} coefficients. By this definition, each member of the set of GSOPs in category IX rendering $N = 1$ ST-SUSY from \mathbf{S}_9 has an effective order of 2. For \mathbf{S}_3 there are $N = 1$ -producing GSOP sets in the same category with projections of effective orders 4 and 2, while the parallel sets in VIII, IV, and III only contain projections with effective orders of 2. For \mathbf{S}_1 , the $N = 1$ -producing GSOP set in IX and in VIII contains projections with effective orders of 8, 4, and 2. \mathbf{S}_1 's solution set in IV contains projections with effective orders of 4 and 2, whereas the solution sets in III contain projections with effective orders of either (i) 4 and 2 or (ii) 2. The sets in classes II and I contain only order 2 projections.

Although we have found nine new solutions for generating $N = 1$ ST-SUSY in free fermionic models, it remains to be shown that these are all physically unique from the standard NAHE solution set of GSOPs and BVs (the category I solution). That is, we

must check for instances when an $N = 1$ model that does not use the standard NAHE solution has equivalent phenomenology to an $N = 1$ model that does. Identities relating partition functions for products of (anti)periodic WS fermions to those for certain products of complex fermions have been derived.^[15,16] These identities will be used to test for possible physical equivalences. If physical equivalences do exist, we suspect they are more likely to involve our new solution sets containing only GSOPs with effective orders of 2.

3. Worldsheet Supercurrent

Local $N = 1$ ST-SUSY in string theory implies not simply the local $N = 1$ worldsheet supersymmetry (WS-SUSY) generated by the internal energy momentum tensor $T(z)^{\text{int}}$ and spin-3/2 supercurrent,

$$T_{\text{F}}^{\text{int}}(z) = i \sum_{J=1}^6 \chi^{3J} \chi^{3J+1} \chi^{3J+2} , \quad (3.1)$$

but an enlargement of this by a spin-1 field, $J(z)$, forming a global $N = 2$ WS-SUSY.^[17] $J(z)$ is the generator of a U(1) Kač-Moody algebra and splits $T_{\text{F}}^{\text{int}}$ into two terms,

$$T_{\text{F}}^{\text{int}} = T_{\text{F}}^{+1} + T_{\text{F}}^{-1} , \quad (3.2)$$

denoted by their respective +1 and -1 U(1) charges. $J(z)$ depends upon the choice of gravitino-generating sector \mathbf{S}_i and takes the general form

$$J(z) = i \mathbf{S}_i^{\text{int}} \cdot \partial_z \mathbf{B} . \quad (3.3)$$

\mathbf{B} is a bosonization (vector) of the complexified internal LM fermionic fields,

$$\exp \{ \pm i B_{K,L} \} = \frac{1}{\sqrt{2}} (\chi^K \pm i \chi^L) , \quad (3.4)$$

and $\mathbf{S}_i^{\text{int}}$ is the internal part of \mathbf{S}_i , with $|\mathbf{S}_i^{\text{int}}|^2 = 3$. Selecting \mathbf{S}_1 ^[18,19] as the gravitino generator results in

$$J_{S_1}(z) = i \partial_z (B_{3,6} + B_{9,12} + B_{15,18}) . \quad (3.5)$$

This U(1) current divides $T_{\text{F}}^{\text{int}}$ into

$$T_{\text{F}}^{-1} = \sum_{J=1,3,5} \frac{i}{\sqrt{2}} \exp(-i B_{3J,3(J+1)}) (\chi^{3J+1} \chi^{3J+2} + i \chi^{3(J+1)+1} \chi^{3(J+1)+2}) , \quad (3.6a)$$

and

$$T_{\text{F}}^{+1} = \sum_{J=1,3,5} \frac{i}{\sqrt{2}} \exp(+i B_{3J,3(J+1)}) (\chi^{3J+1} \chi^{3J+2} - i \chi^{3(J+1)+1} \chi^{3(J+1)+2}) . \quad (3.6b)$$

The comparable expressions for $T_F^{\pm 1}$'s (written in a diagonalized basis of fields) become a bit more complicated when either \mathbf{S}_3 or \mathbf{S}_9 is the gravitino generator. Nevertheless, \mathbf{S}_1 is not the only choice of gravitino sector consistent with the $N = 2$ WS-SUSY. For example, in the initial non-diagonal $\{\chi^i\}$ basis, the \mathbf{S}_3 automorphism arises from the combination of the non-trivial inner automorphisms,

$$(\chi^{3J}, \chi^{3J+1}, \chi^{3J+2}) \rightarrow (\chi^{3J}, -\chi^{3J+1}, -\chi^{3J+2}) \quad \text{for } J = 1, 3, 5, 6 \quad (3.7)$$

with the outer automorphism,

$$(\chi^{3J}, \chi^{3J+1}, \chi^{3J+2}) \leftrightarrow (\chi^{3(J+1)}, \chi^{3(J+1)+1}, \chi^{3(J+1)+2}) \quad \text{for } J = 1, 3. \quad (3.8)$$

Note that T_F^{int} is invariant under these transformations, as is required by (3.1). The eigenstates of the automorphism (with eigenvalues in LHS parenthesis) are:

$$(-1) : u^{3,6} = \frac{1}{\sqrt{2}}(-\chi^3 + \chi^6) \quad (3.9a)$$

$$(+1) : v^{3,6} = \frac{1}{\sqrt{2}}(\chi^3 + \chi^6) \quad (3.9b)$$

$$(+i) : u^{4,7} = \frac{-1}{\sqrt{2}}(\chi^7 - i\chi^4) \quad (3.9c)$$

$$(-i) : u^{5,8} = \frac{-1}{\sqrt{2}}(\chi^8 + i\chi^5) \quad (3.9d)$$

$$(-1) : u^{9,12} = \frac{1}{\sqrt{2}}(-\chi^9 + \chi^{12}) \quad (3.9e)$$

$$(+1) : v^{9,12} = \frac{1}{\sqrt{2}}(\chi^9 + \chi^{12}) \quad (3.9f)$$

$$(+i) : u^{10,13} = \frac{-1}{\sqrt{2}}(\chi^{10} - i\chi^{13}) \quad (3.9g)$$

$$(-i) : u^{12,14} = \frac{-1}{\sqrt{2}}(\chi^{12} + i\chi^{14}) \quad (3.9h)$$

$$(+1) : \frac{1}{\sqrt{2}}(\chi^{15} \pm i\chi^{18}) \quad (3.9i)$$

$$(-1) : \chi^{16}, \chi^{17}, \chi^{19}, \chi^{20}.$$

and complex conjugates of $u^{4,7}$, $u^{5,8}$, $u^{10,13}$, and $u^{11,14}$. We can bosonize these fermion eigenstates as follows:

$$\exp \{ (-)i\pi B_{I,I+3} \} = u^{I,I+3(*)} \quad \text{for } I = 4, 5, 10, 12 \quad (3.10a)$$

$$\exp \{ \pm i\pi B_{3,6(-1)} \} = \frac{1}{\sqrt{2}}(u^{3,6} \pm iu^{9,12}) \quad (3.10b)$$

$$\exp \{ \pm i\pi B_{3,6(+1)} \} = \frac{1}{\sqrt{2}}(v^{3,6} \pm iv^{9,12}) \quad (3.10c)$$

$$\exp \{ \pm i\pi B_{15,18} \} = \frac{1}{\sqrt{2}}(\chi^{15} \pm i\chi^{18}). \quad (3.10d)$$

The $U(1)$ current and ± 1 -charged supercurrents then appear as,

$$J_{S_3} = i\partial \left(\mathbf{B}_{3,6(+)} - \frac{1}{2}\mathbf{B}_{4,7} + \frac{1}{2}\mathbf{B}_{5,8} + \mathbf{B}_{9,12(+)} - \frac{1}{2}\mathbf{B}_{10,13} + \frac{1}{2}\mathbf{B}_{11,14} + \mathbf{B}_{15,18} \right) \quad (3.11)$$

$$\begin{aligned} T_F^- = & \frac{i}{2} \exp\{-i\mathbf{B}_{3,6(-)}\} \left(\exp\{+i\mathbf{B}_{4,7}\} \exp\{-i\mathbf{B}_{5,8}\} + i \exp\{+i\mathbf{B}_{10,13}\} \exp\{-i\mathbf{B}_{11,14}\} \right) \\ & + \frac{i}{2} \exp\{+i\mathbf{B}_{3,6(-)}\} \left(\exp\{+i\mathbf{B}_{4,7}\} \exp\{-i\mathbf{B}_{5,8}\} - i \exp\{+i\mathbf{B}_{10,13}\} \exp\{-i\mathbf{B}_{11,14}\} \right) \\ & + \frac{i}{2} \exp\{-i\mathbf{B}_{3,6(+)}\} \left(\exp\{-i\mathbf{B}_{4,7}\} \exp\{-i\mathbf{B}_{5,8}\} + i \exp\{-i\mathbf{B}_{10,13}\} \exp\{-i\mathbf{B}_{11,14}\} \right) \\ & + \frac{i}{2} \exp\{-i\mathbf{B}_{3,6(+)}\} \left(\exp\{+i\mathbf{B}_{4,7}\} \exp\{+i\mathbf{B}_{5,8}\} + i \exp\{+i\mathbf{B}_{10,13}\} \exp\{+i\mathbf{B}_{11,14}\} \right) \\ & + \frac{i}{2} \exp\{-i\mathbf{B}_{18,21}\} (\chi^{16}\chi^{17} + i\chi^{19}\chi^{20}) . \end{aligned} \quad (3.12)$$

$$T_F^+ = -(T_F^+)^* . \quad (3.13)$$

4. Comments

We have completely classified the ways by which the number of spacetime supersymmetries in heterotic free fermionic strings may be reduced from $N = 4$ to the phenomenologically preferred $N = 1$. This means that the set of LM boundary vectors in any free fermionic model with $N = 1$ ST-SUSY must be reproducible from a combination of one of the three gravitino sectors, \mathbf{S}_1 , \mathbf{S}_3 , or \mathbf{S}_9 , with one of our accompanying $N = 1$ reduction sets. The only variations from our LM BV sets that true $N = 1$ models could have are (1) trivial reordering of the BV worldsheet fermions, or (2) trivial phase changes by minus signs. Neither variation leads to physically distinct models. The latter variation corresponds to either (1) using one of the odd multiples of \mathbf{S}_i (of the same order as \mathbf{S}_i), rather than \mathbf{S}_i itself, to generate the surviving gravitino, or (2) (if applicable) using boundary vector components that commute with $S(4, 1, 2, 1)$ rather than with $S(4, 1, 1, 1)$.

To this date, only the gravitino generator \mathbf{S}_1 has been used in actual $N = 1$ free fermionic models. Reduction to $N = 1$ ST-SUSY has been accomplished through use of the LM NAHE set discussed previously. Thus, our new results should be especially useful for model building when the NAHE set may be inconsistent with other properties specifically desired of a model. This appears to be the situation with regard to current searches for consistent three generation $SO(10)$ level-2 models. Initial results of this search were discussed in refs. [20] and [21]. Initial attempts to simultaneously produce $N = 1$ SUSY and a three generation $SO(10)$ level-2 grand unified theory were thought to be successful using \mathbf{S}_1 , the LM NAHE set, and one or two additional BVs containing some non-integer components. However, later it was discovered that the extra non-integer BVs required

did not correspond to proper $SU(2)^6$ automorphisms and, therefore, produced additional sectors containing tachyonic spacetime fermions, making the models inconsistent.

The next step in classification of free fermionic models with potentially good phenomenology is to test for physical equivalences between our new $N = 1$ ST-SUSY solutions and the standard NAHE solution. Following this, we will investigate which of our physically unique LM $N = 1$ ST-SUSY solutions may be consistent with three generations. While the research presented herein was underway, a paper exploring this issue appeared. Ref. [19] shows that the number of generations in an $N = 1$ ST-SUSY model is related to the index of the underlying $N = 2$ internal superconformal field theory. Determination of the index requires knowledge of both the LM and RM components of the BVs in a model. Thus, this technique cannot be applied directly to our $N = 1$ solutions since they have only LM components. However, slightly modifying this technique, we will investigate in upcoming papers^[22] which (if any) of our sets of left-movers have the *potential*, when paired with appropriate right-movers, to yield exactly three chiral generations for a given gauge group, in particular for $SO(10)$ level-2.

Acknowledgements

Our sincere thanks to Henry Tye, Jorge Lopez, and Kajia Yuan for helpful discussions. We also note that unpublished work on classification of free fermionic spacetime supersymmetries has been done by the authors of ref. [12]. The contents of Tables 1 and 2 and some of the boundary vector components in Table 3 were first presented in [12].

TABLE 1. The α in $-\exp\{i\pi\alpha\}$ eigenvalues of length- n cyclic permutations combined with degree of freedom θ from $SU(2)$ inner automorphism.

n	α 's for J_3	α 's for J_+, J_- (for J_1, J_2 if $\theta = 0, 1$)
1	1	$1 - \theta$
2	0, 1	$-\frac{\theta}{2}, 1 - \frac{\theta}{2}$
3	$\pm\frac{1}{3}, 1$	$\pm\frac{1}{3} - \frac{\theta}{3}, 1 - \frac{\theta}{3}$
4	0, $\pm\frac{1}{2}, 1$	$-\frac{\theta}{4}, \pm\frac{1}{2} - \frac{\theta}{4}, 1 - \frac{\theta}{4}$
5	$\pm\frac{1}{5}, \pm\frac{3}{5}, 1$	$\pm\frac{1}{5} - \frac{\theta}{5}, \pm\frac{3}{5} - \frac{\theta}{5}, 1 - \frac{\theta}{5}$
6	0, $\pm\frac{1}{3}, \pm\frac{2}{3}, 1$	$-\frac{\theta}{6}, \pm\frac{1}{3} - \frac{\theta}{6}, \pm\frac{2}{3} - \frac{\theta}{6}, 1 - \frac{\theta}{6}$

TABLE 2. Representatives, \mathbf{S}_i , of the eleven distinct classes of left-moving boundary vectors that can yield massless fermions.

Classes are defined by lengths of non-disjoint cycles in the $SU(2)^6$ permutation upon which a boundary vector is based. Only \mathbf{S}_1 , \mathbf{S}_3 , \mathbf{S}_5 , \mathbf{S}_7 , \mathbf{S}_9 , and \mathbf{S}_{10} have an even number of periodic components (1's). They are the only potential gravitino generators.

The first two components of each \mathbf{S}_i are the transverse spacetime components $\psi^{1,2}$ for lightcone gauge. The boundary vector components for each non-disjoint cycle of length- n_k in \mathbf{S}_i are enclosed in parentheses. The components on the left-hand side of the semi-colons correspond to the n_k eigenvalues for the J_3 fermions in each group. The components in the first (second) half of the right-hand side of the semicolon correspond to the eigenvalues for the J_1 (J_2) fermions. A “^” denotes an boundary vector component for a complex fermion. For each complex fermion eigenstate with a complex eigenvalue (non-integer boundary component), there is also a complex conjugate eigenstate.

Class	Representative Massless Boundary Vector
$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$	$\mathbf{S}_1 = \{1, 1 \quad (1; 0, 0) \quad (1; 0, 0) \quad (1; 0, 0) \quad (1; 0, 0) \quad (1; 0, 0) \quad (1; 0, 0)\}$
$2 \cdot 1 \cdot 1 \cdot 1 \cdot 1$	$\mathbf{S}_2 = \{1, 1 \quad (0, 1; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}) \quad (1; 0, 0) \quad (1; 0, 0) \quad (1; 0, 0) \quad (1; 0, 0)\}$
$2 \cdot 2 \cdot 1 \cdot 1$	$\mathbf{S}_3 = \{1, 1 \quad (0, 1; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}) \quad (0, 1; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}) \quad (1; 0, 0) \quad (1; 0, 0)\}$
$2 \cdot 2 \cdot 2$	$\mathbf{S}_4 = \{1, 1 \quad (0, 1; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}) \quad (0, 1; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}) \quad (0, 1; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2})\}$
$3 \cdot 1 \cdot 1 \cdot 1$	$\mathbf{S}_5 = \{1, 1 \quad (\frac{\hat{1}}{3}, 1; -\frac{\hat{2}}{3}, 0, 0, \frac{\hat{2}}{3}) \quad (1; 0, 0) \quad (1; 0, 0) \quad (1; 0, 0)\}$
$3 \cdot 2 \cdot 1$	$\mathbf{S}_6 = \{1, 1 \quad (\frac{\hat{1}}{3}, 1; -\frac{\hat{2}}{3}, 0, 0, \frac{\hat{2}}{3}) \quad (0, 1; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}) \quad (1; 0, 0)\}$
$3 \cdot 3$	$\mathbf{S}_7 = \{1, 1 \quad (\frac{\hat{1}}{3}, 1; -\frac{\hat{2}}{3}, 0, 0, \frac{\hat{2}}{3}) \quad (\frac{\hat{1}}{3}, 1; -\frac{\hat{2}}{3}, 0, 0, \frac{\hat{2}}{3})\}$
$4 \cdot 1 \cdot 1$	$\mathbf{S}_8 = \{1, 1 \quad (0, \frac{\hat{1}}{2}, 1; -\frac{\hat{3}}{4}, -\frac{\hat{1}}{4}, \frac{\hat{1}}{4}, \frac{\hat{3}}{4}) \quad (1; 0, 0) \quad (1; 0, 0)\}$
$4 \cdot 2$	$\mathbf{S}_9 = \{1, 1 \quad (0, \frac{\hat{1}}{2}, 1; -\frac{\hat{3}}{4}, -\frac{\hat{1}}{4}, \frac{\hat{1}}{4}, \frac{\hat{3}}{4}) \quad (0, 1; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2})\}$
$5 \cdot 1$	$\mathbf{S}_{10} = \{1, 1 \quad (\frac{\hat{1}}{5}, \frac{\hat{3}}{5}, 1; -\frac{\hat{4}}{5}, -\frac{\hat{2}}{5}, 0, 0, \frac{\hat{2}}{5}, \frac{\hat{4}}{5}) \quad (1; 0, 0)\}$
6	$\mathbf{S}_{11} = \{1, 1 \quad (0, \frac{\hat{1}}{3}, \frac{\hat{2}}{3}, 1; -\frac{\hat{5}}{6}, -\frac{\hat{1}}{2}, -\frac{\hat{1}}{6}, \frac{\hat{1}}{6}, \frac{\hat{1}}{2}, \frac{\hat{5}}{6})\}$

Order $N_{S_i} = 2, 4, 4, 4, 6, 12, 6, 8, 8, 10, 12$ for $i = 1$ to 11, respectively.

TABLE 3. Sets of commuting boundary vectors and classes of GSO projections for components of gravitino-generating basis vectors.

Listed below are all possible sets of mutually commuting boundary vectors for the worldsheet supersymmetric sector of heterotic free fermionic strings. Each element of length $3 \sum n_k = 3n_*$ (with complex components counting double) in a set of commuting boundary vectors is derived from a cyclic permutation that is a power of the non-disjoint cycle of length n_* from the set's first boundary vector. In the notation below, the boundary vectors designated as $S(n_*, y, z, w)$ are classified by: (1) the total length of their permutations, n_* , (2) the power of the cyclic permutations used, y , (3) the various ways of dividing their eigenvalues among the J_1 's and J_2 's, numbered by z , and (4) the choice of inner automorphism, specified by w .

In each commuting set, the distinct length- $3n_*$ boundary vectors that could potentially appear in one (or more) of the six gravitino sectors are marked with a “ \rightarrow ”. All boundary vectors are classified according to their contributions to the GSO projections operating on these “arrowed” boundary vectors. The important properties defining a boundary vector's contributions to such GSO operators are: (i) the boundary vector's components that correspond to the “arrowed” vector's periodic components, referred to as ***F*** coefficients, and (ii) half of the dot product of the boundary vector with the “arrowed” vector. Since we can shown that varying the division of the components among the J_1 's and J_2 's does not alter the set of boundary vectors composing a class, GSO data is given only for $z = 1$.

Cycles of Lengths n_i	Order N		Designation	Boundary Vector Components
1	2	\rightarrow	$S(1, 1, 1, 1)$	$= (1; 0, 0)$
			$S(1, 1, 1, 2)$	$= (0; 1, 0)$
			$S(1, 1, 1, 3)$	$= (0; 0, 1)$
			$S(1, 1, 1, 4)$	$= (1; 1, 1)$
GSO projection classes for $S(1, 1, 1, 1)$ contribution to gravitino sectors S_1, S_3, S_5, S_7				
class (1.1.1.1):	<i>F</i> coef = 1	and $\frac{1}{2}\text{dp} = \frac{1}{4}$		(mod 1)
	$\{S(1, 1, 1, 1), S(1, 1, 1, 4)\}$			
class (1.1.1.2):	<i>F</i> coef = 0	and $\frac{1}{2}\text{dp} = 0$		(mod 1)
	$\{S(1, 1, 1, 2), S(1, 1, 1, 3)\}$			

Cycles of Lengths n_i	Order N	Designation	Boundary Vector Components
2	4	$\rightarrow S(2, 1, 1, 1)$	$= (0, 1; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2})$
		$S(2, 1, 1, 2)$	$= (1, 0; \frac{\hat{1}}{2}, \frac{\hat{1}}{2})$
		$S(2, 1, 1, 3)$	$= (1, 0; -\frac{\hat{1}}{2}, -\frac{\hat{1}}{2})$
		$S(2, 1, 1, 4)$	$= (0, 1; \frac{\hat{1}}{2}, -\frac{\hat{1}}{2})$
$1 \cdot 1$	2	$\rightarrow S(2, 2, 1, 1)$	$= (1, 1; \hat{0}, \hat{0})$
		$\rightarrow S(2, 2, 1, 2)$	$= (0, 0; \hat{1}, \hat{0})$
		$S(2, 2, 1, 3)$	$= (0, 0; \hat{0}, \hat{1})$
		$S(2, 2, 1, 4)$	$= (1, 1; \hat{1}, \hat{1})$

GSO projection classes for $S(2, 1, 1, 1)$ contribution to gravitino sectors $\mathbf{S}_3, \mathbf{S}_9$

class (2.2.1.1): \mathbf{F} coef = 1 and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{\frac{1}{2}}$
 $\{S(2, 1, 1, 1), S(2, 1, 1, 4)\}$

class (2.2.1.2): \mathbf{F} coef = 1 and $\frac{1}{2}\text{dp} = \frac{1}{4} \pmod{\frac{1}{2}}$
 $\{S(2, 2, 1, 1), S(2, 2, 1, 4)\}$

class (2.2.1.3): \mathbf{F} coef = 0 and $\frac{1}{2}\text{dp} = 0 \pmod{\frac{1}{2}}$
 $\{S(2, 1, 1, 2), S(2, 1, 1, 3)\}$

class (2.2.1.4): \mathbf{F} coef = 0 and $\frac{1}{2}\text{dp} = -\frac{1}{4} \pmod{\frac{1}{2}}$
 $\{S(2, 2, 1, 2), S(2, 2, 1, 3)\}$

GSO projection classes for $S(2, 2, 1, 1)$ contribution to gravitino sectors $\mathbf{S}_1, \mathbf{S}_3, \mathbf{S}_5$

class (1.2.1.1): \mathbf{F} coef = 1, 1 and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{1}$
 $\{S(2, 2, 1, 1), S(2, 2, 1, 4)\}$

class (1.2.1.2): \mathbf{F} coef = 1, 0 and $\frac{1}{2}\text{dp} = \frac{1}{4} \pmod{1}$
 $\{S(2, 1, 1, 2), S(2, 1, 1, 3)\}$

class (1.2.1.3): \mathbf{F} coef = 0, 1 and $\frac{1}{2}\text{dp} = \frac{1}{4} \pmod{1}$
 $\{S(2, 1, 1, 1), S(2, 1, 1, 4)\}$

class (1.2.1.4): \mathbf{F} coef = 0, 0 and $\frac{1}{2}\text{dp} = 0 \pmod{1}$
 $\{S(2, 2, 1, 2), S(2, 2, 1, 3)\}$

GSO projection classes for $S(2, 2, 1, 2)$ contribution to gravitino sectors $\mathbf{S}_1, \mathbf{S}_3, \mathbf{S}_5$

class (1.2.2.1): \mathbf{F} coef = $\hat{1}$ and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{1}$
 $\{S(2, 2, 1, 2), S(2, 2, 1, 4)\}$

class (1.2.2.2): \mathbf{F} coef = $\hat{0}$ and $\frac{1}{2}\text{dp} = 0 \pmod{1}$
 $\{S(2, 2, 1, 1), S(2, 2, 1, 3)\}$

class (1.2.2.3): \mathbf{F} coef = $\frac{\hat{1}}{2}$ and $\frac{1}{2}\text{dp} = \frac{1}{4} \pmod{1}$
 $\{S(2, 1, 1, 2), S(2, 1, 1, 4)\}$

class (1.2.2.4): \mathbf{F} coef = $-\frac{\hat{1}}{2}$ and $\frac{1}{2}\text{dp} = -\frac{1}{4} \pmod{1}$
 $\{S(2, 1, 1, 1), S(2, 1, 1, 3)\}$

Cycles of Lengths n_i	Order N	Designation	Boundary Vector Components
3	6	$\rightarrow S(3, 1, 1, 1) =$	$(\frac{\hat{1}}{3}, 1; -\frac{\hat{2}}{3}, 0, 0, \frac{\hat{2}}{3})$
		$S(3, 1, 1, 2) =$	$(-\frac{\hat{2}}{3}, 0; \frac{\hat{1}}{3}, 1, 0, \frac{\hat{2}}{3})$
		$S(3, 1, 1, 3) =$	$(-\frac{\hat{2}}{3}, 0; -\frac{\hat{2}}{3}, 0, 1, -\frac{\hat{1}}{3})$
		$S(3, 1, 1, 4) =$	$(\frac{\hat{1}}{3}, 1; \frac{\hat{1}}{3}, 1, 1, -\frac{\hat{1}}{3})$
		$S(3, 2, 1, 1) =$	$(-\frac{\hat{1}}{3}, 1; -\frac{\hat{1}}{3}, 1, 1, \frac{\hat{1}}{3})$
		$S(3, 2, 1, 2) =$	$(\frac{\hat{2}}{3}, 0; \frac{\hat{2}}{3}, 0, 1, \frac{\hat{1}}{3})$
		$S(3, 2, 1, 3) =$	$(\frac{\hat{2}}{3}, 0; -\frac{\hat{1}}{3}, 1, 0, -\frac{\hat{2}}{3})$
		$S(3, 2, 1, 4) =$	$(-\frac{\hat{1}}{3}, 1; \frac{\hat{2}}{3}, 0, 0, -\frac{\hat{2}}{3})$
$1 \cdot 1 \cdot 1$	2	$\rightarrow S(3, 3, 1, 1) =$	$(\hat{1}, 1; \hat{0}, 0, 0, \hat{0})$
		$S(3, 3, 1, 2) =$	$(\hat{0}, 0; \hat{1}, 1, 0, \hat{0})$
		$S(3, 3, 1, 3) =$	$(\hat{0}, 0; \hat{0}, 0, 1, \hat{1})$
		$S(3, 3, 1, 4) =$	$(\hat{1}, 1; \hat{1}, 1, 1, \hat{1})$

GSO projection classes for $S(3, 1, 1, 1)$ contribution to gravitino sectors $\mathbf{S}_5, \mathbf{S}_7$

class (3.3.1.1): \mathbf{F} coef = 1 and $\frac{1}{2}\text{dp} = \frac{3}{4} \pmod{\frac{1}{3}}$
 $\{S(3, 1, 1, 1), S(3, 1, 1, 4), S(3, 2, 1, 1), S(3, 2, 1, 4), S(3, 3, 1, 1), S(3, 3, 1, 4)\}$

class (3.3.1.2): \mathbf{F} coef = 0 and $\frac{1}{2}\text{dp} = 0 \pmod{\frac{1}{3}}$
 $\{S(3, 1, 1, 2), S(3, 1, 1, 3), S(3, 2, 1, 2), S(3, 2, 1, 3), S(3, 3, 1, 2), S(3, 3, 1, 3)\}$

GSO projection classes for $S(3, 3, 1, 1)$ contribution to gravitino sectors $\mathbf{S}_1, \mathbf{S}_5$

class (1.3.1.1): \mathbf{F} coef = $\hat{1}, 1$ and $\frac{1}{2}\text{dp} = \frac{3}{4} \pmod{1}$
 $\{S(3, 3, 1, 1), S(3, 3, 1, 4)\}$

class (1.3.1.2): \mathbf{F} coef = $\hat{0}, 0$ and $\frac{1}{2}\text{dp} = 0 \pmod{1}$
 $\{S(3, 3, 1, 2), S(3, 3, 1, 3)\}$

class (1.3.1.3): \mathbf{F} coef = $\frac{\hat{1}}{3}, 1$ and $\frac{1}{2}\text{dp} = \frac{5}{12} \pmod{1}$
 $\{S(3, 1, 1, 1), S(3, 1, 1, 4)\}$

class (1.3.1.4): \mathbf{F} coef = $-\frac{\hat{1}}{3}, 1$ and $\frac{1}{2}\text{dp} = \frac{1}{12} \pmod{1}$
 $\{S(3, 2, 1, 1), S(3, 2, 1, 4)\}$

class (1.3.1.5): \mathbf{F} coef = $\frac{\hat{2}}{3}, 0$ and $\frac{1}{2}\text{dp} = \frac{1}{3} \pmod{1}$
 $\{S(3, 2, 1, 2), S(3, 2, 1, 3)\}$

class (1.3.1.6): \mathbf{F} coef = $-\frac{\hat{2}}{3}, 0$ and $\frac{1}{2}\text{dp} = -\frac{1}{3} \pmod{1}$
 $\{S(3, 1, 1, 2), S(3, 1, 1, 3)\}$

Cycles of Lengths n_i	Order N	Designation	Boundary Vector Components
4	8	→ $S(4, 1, 1, 1)$	$= (\frac{\hat{1}}{2}, 0, 1; -\frac{\hat{3}}{4}, -\frac{\hat{1}}{4}, \frac{\hat{1}}{4}, \frac{\hat{3}}{4})$
		$S(4, 1, 1, 2)$	$= (-\frac{\hat{1}}{2}, 1, 0; \frac{\hat{1}}{4}, \frac{\hat{3}}{4}, \frac{\hat{1}}{4}, \frac{\hat{3}}{4})$
		$S(4, 1, 1, 3)$	$= (-\frac{\hat{1}}{2}, 1, 0; -\frac{\hat{3}}{4}, -\frac{\hat{1}}{4}, -\frac{\hat{3}}{4}, -\frac{\hat{1}}{4})$
		$S(4, 1, 1, 4)$	$= (\frac{\hat{1}}{2}, 0, 1; \frac{\hat{1}}{4}, \frac{\hat{3}}{4}, -\frac{\hat{3}}{4}, -\frac{\hat{1}}{4})$
		$S(4, 3, 1, 1)$	$= (-\frac{\hat{1}}{2}, 0, 1; -\frac{\hat{1}}{4}, -\frac{\hat{3}}{4}, \frac{\hat{3}}{4}, \frac{\hat{1}}{4})$
		$S(4, 3, 1, 2)$	$= (\frac{\hat{1}}{2}, 1, 0; \frac{\hat{3}}{4}, \frac{\hat{1}}{4}, \frac{\hat{3}}{4}, \frac{\hat{1}}{4})$
		$S(4, 3, 1, 3)$	$= (\frac{\hat{1}}{2}, 1, 0; -\frac{\hat{1}}{4}, -\frac{\hat{3}}{4}, -\frac{\hat{1}}{4}, -\frac{\hat{3}}{4})$
		$S(4, 3, 1, 4)$	$= (-\frac{\hat{1}}{2}, 0, 1; \frac{\hat{3}}{4}, \frac{\hat{1}}{4}, -\frac{\hat{1}}{4}, -\frac{\hat{3}}{4})$
2 · 2	4	→ $S(4, 2, 1, 1)$	$= (\hat{0}, 1, 1; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}, -\frac{\hat{1}}{2}, \frac{\hat{1}}{2})$
		→ $S(4, 2, 1, 2)$	$= (\hat{1}, 0, 0; \frac{\hat{1}}{2}, -\frac{\hat{1}}{2}, -\frac{\hat{1}}{2}, \frac{\hat{1}}{2})$
		$S(4, 2, 1, 3)$	$= (\hat{1}, 0, 0; -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}, \frac{\hat{1}}{2}, -\frac{\hat{1}}{2})$
		$S(4, 2, 1, 4)$	$= (\hat{0}, 1, 1; \frac{\hat{1}}{2}, -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}, -\frac{\hat{1}}{2})$
1 · 1 · 1 · 1	2	→ $S(4, 4, 1, 1)$	$= (\hat{1}, 1, 1; \hat{0}, \hat{0}, \hat{0}, \hat{0})$
		→ $S(4, 4, 1, 2)$	$= (\hat{0}, 0, 0; \hat{1}, \hat{1}, \hat{0}, \hat{0})$
		$S(4, 4, 1, 3)$	$= (\hat{0}, 0, 0; \hat{0}, \hat{0}, \hat{1}, \hat{1})$
		$S(4, 4, 1, 4)$	$= (\hat{1}, 1, 1; \hat{1}, \hat{1}, \hat{1}, \hat{1})$

GSO projection classes for $S(4, 1, 1, 1)$ contribution to gravitino sector \mathbf{S}_9

class (4.4.1.1): \mathbf{F} coef = 1 and $\frac{1}{2}\text{dp} = 1 \pmod{\frac{1}{4}}$
 $\{S(4, 1, 1, 1), S(4, 1, 1, 4), S(4, 3, 1, 1), S(4, 3, 1, 4)\}$

class (4.4.1.2): \mathbf{F} coef = 1 and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{\frac{1}{4}}$
 $\{S(4, 2, 1, 1), S(4, 2, 1, 4), S(4, 4, 1, 1), S(4, 4, 1, 4)\}$

class (4.4.1.3): \mathbf{F} coef = 0 and $\frac{1}{2}\text{dp} = 0 \pmod{\frac{1}{4}}$
 $\{S(4, 1, 1, 2), S(4, 1, 1, 3), S(4, 3, 1, 2), S(4, 3, 1, 3)\}$

class (4.4.1.4): \mathbf{F} coef = 0 and $\frac{1}{2}\text{dp} = \frac{1}{4} \pmod{\frac{1}{4}}$
 $\{S(4, 2, 1, 2), S(4, 2, 1, 3), S(4, 4, 1, 2), S(4, 4, 1, 3)\}$

GSO projection classes for $S(4, 2, 1, 1)$ contribution to gravitino sector \mathbf{S}_3

class (2.4.1.1): \mathbf{F} coef = 1, 1 and $\frac{1}{2}\text{dp} = 1 \pmod{\frac{1}{2}}$
 $\{S(4, 2, 1, 1), S(4, 2, 1, 4), S(4, 4, 1, 1), S(4, 4, 1, 4)\}$

class (2.4.1.2): \mathbf{F} coef = 1, 0 and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{\frac{1}{2}}$
 $\{S(4, 1, 1, 2), S(4, 1, 1, 3), S(4, 3, 1, 2), S(4, 3, 1, 3)\}$

class (2.4.1.3): \mathbf{F} coef = 0, 1 and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{\frac{1}{2}}$
 $\{S(4, 1, 1, 1), S(4, 1, 1, 4), S(4, 3, 1, 1), S(4, 3, 1, 4)\}$

class (2.4.1.4): \mathbf{F} coef = 0, 0 and $\frac{1}{2}\text{dp} = 0 \pmod{\frac{1}{2}}$
 $\{S(4, 2, 1, 2), S(4, 2, 1, 3), S(4, 4, 1, 2), S(4, 4, 1, 3)\}$

GSO projection classes for $S(4, 2, 1, 2)$ contribution to gravitino sector \mathbf{S}_3

class (2.4.2.1): \mathbf{F} coef = $\hat{1}$ and $\frac{1}{2}\text{dp} = 1 \pmod{\frac{1}{2}}$
 $\{S(4, 2, 1, 2), S(4, 2, 1, 3), S(4, 4, 1, 1), S(4, 4, 1, 4)\}$

class (2.4.2.2): \mathbf{F} coef = $\hat{0}$ and $\frac{1}{2}\text{dp} = 0 \pmod{\frac{1}{2}}$
 $\{S(4, 2, 1, 1), S(4, 2, 1, 4), S(4, 4, 1, 2), S(4, 4, 1, 3)\}$

class (2.4.2.3): \mathbf{F} coef = $\frac{\hat{1}}{2}$ and $\frac{1}{2}\text{dp} = \frac{1}{4} \pmod{\frac{1}{2}}$
 $\{S(4, 1, 1, 1), S(4, 1, 1, 4), S(4, 3, 1, 2), S(4, 3, 1, 3)\}$

class (2.4.2.4): \mathbf{F} coef = $-\frac{\hat{1}}{2}$ and $\frac{1}{2}\text{dp} = -\frac{1}{4} \pmod{\frac{1}{2}}$
 $\{S(4, 1, 1, 2), S(4, 1, 1, 3), S(4, 3, 1, 1), S(4, 3, 1, 4)\}$

GSO projection classes for $S(4, 4, 1, 1)$ contribution to gravitino sector \mathbf{S}_1

class (1.4.1.1): \mathbf{F} coef = $\hat{1}$, 1, 1 and $\frac{1}{2}\text{dp} = 1 \pmod{1}$
 $\{S(4, 4, 1, 1), S(4, 4, 1, 4)\}$

class (1.4.1.2): \mathbf{F} coef = $\hat{1}$, 0, 0 and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{1}$
 $\{S(4, 2, 1, 2), S(4, 2, 1, 3)\}$

class (1.4.1.3): \mathbf{F} coef = $\hat{0}$, 1, 1 and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{1}$
 $\{S(4, 2, 1, 1), S(4, 2, 1, 4)\}$

class (1.4.1.4): \mathbf{F} coef = $\hat{0}$, 0, 0 and $\frac{1}{2}\text{dp} = 0 \pmod{1}$
 $\{S(4, 4, 1, 2), S(4, 4, 1, 3)\}$

class (1.4.1.5): \mathbf{F} coef = $\frac{\hat{1}}{2}$, 1, 0 and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{1}$
 $\{S(4, 3, 1, 2), S(4, 3, 1, 3)\}$

class (1.4.1.6): \mathbf{F} coef = $-\frac{\hat{1}}{2}$, 1, 0 and $\frac{1}{2}\text{dp} = 0 \pmod{1}$
 $\{S(4, 1, 1, 2), S(4, 1, 1, 3)\}$

$$\text{class (1.4.1.7): } \mathbf{F} \text{ coef} = \frac{\hat{1}}{2}, 0, 1 \text{ and } \frac{1}{2}\text{dp} = \frac{1}{2} \pmod{1} \\ \{S(4, 1, 1, 1), S(4, 1, 1, 4)\}$$

$$\text{class (1.4.1.8): } \mathbf{F} \text{ coef} = -\frac{\hat{1}}{2}, 0, 1 \text{ and } \frac{1}{2}\text{dp} = 0 \pmod{1} \\ \{S(4, 3, 1, 1), S(4, 3, 1, 4)\}$$

GSO projection classes for $S(4, 4, 1, 2)$ contribution to gravitino sector \mathbf{S}_1

$$\text{class (1.4.2.1): } \mathbf{F} \text{ coef} = \hat{1}, \hat{1} \text{ and } \frac{1}{2}\text{dp} = 1 \pmod{1} \\ \{S(4, 4, 1, 2), S(4, 4, 1, 4)\}$$

$$\text{class (1.4.2.2): } \mathbf{F} \text{ coef} = \hat{0}, \hat{0} \text{ and } \frac{1}{2}\text{dp} = 0 \pmod{1} \\ \{S(4, 4, 1, 1), S(4, 4, 1, 3)\}$$

$$\text{class (1.4.2.3): } \mathbf{F} \text{ coef} = \frac{\hat{3}}{4}, \frac{\hat{1}}{4} \text{ and } \frac{1}{2}\text{dp} = \frac{1}{2} \pmod{1} \\ \{S(4, 3, 1, 2), S(4, 3, 1, 4)\}$$

$$\text{class (1.4.2.4): } \mathbf{F} \text{ coef} = -\frac{\hat{3}}{4}, -\frac{\hat{1}}{4} \text{ and } \frac{1}{2}\text{dp} = -\frac{1}{2} \pmod{1} \\ \{S(4, 1, 1, 1), S(4, 1, 1, 3)\}$$

$$\text{class (1.4.2.5): } \mathbf{F} \text{ coef} = \frac{\hat{1}}{2}, -\frac{\hat{1}}{2} \text{ and } \frac{1}{2}\text{dp} = 0 \pmod{1} \\ \{S(4, 2, 1, 2), S(4, 2, 1, 4)\}$$

$$\text{class (1.4.2.6): } \mathbf{F} \text{ coef} = -\frac{\hat{1}}{2}, \frac{\hat{1}}{2} \text{ and } \frac{1}{2}\text{dp} = 0 \pmod{1} \\ \{S(4, 2, 1, 1), S(4, 2, 1, 3)\}$$

$$\text{class (1.4.2.7): } \mathbf{F} \text{ coef} = \frac{\hat{1}}{4}, \frac{\hat{3}}{4} \text{ and } \frac{1}{2}\text{dp} = \frac{1}{2} \pmod{1} \\ \{S(4, 1, 1, 2), S(4, 1, 1, 4)\}$$

$$\text{class (1.4.2.8): } \mathbf{F} \text{ coef} = -\frac{\hat{1}}{4}, -\frac{\hat{3}}{4} \text{ and } \frac{1}{2}\text{dp} = -\frac{1}{2} \pmod{1} \\ \{S(4, 3, 1, 1), S(4, 3, 1, 3)\}$$

Cycles of Lengths n_i	Order N	Designation	Boundary Vector Components
4	8	$S(4, 1, 2, 1) =$	$(\frac{\hat{1}}{2}, 0, 1; -\frac{\hat{3}}{4}, \frac{\hat{1}}{4}, -\frac{\hat{1}}{4}, \frac{\hat{3}}{4})$
		$S(4, 1, 2, 2) =$	$(-\frac{\hat{1}}{2}, 1, 0; \frac{\hat{1}}{4}, -\frac{\hat{3}}{4}, -\frac{\hat{1}}{4}, \frac{\hat{3}}{4})$
		$S(4, 1, 2, 3) =$	$(-\frac{\hat{1}}{2}, 1, 0; -\frac{\hat{3}}{4}, \frac{\hat{1}}{4}, \frac{\hat{3}}{4}, -\frac{\hat{1}}{4})$
		$S(4, 1, 2, 4) =$	$(\frac{\hat{1}}{2}, 0, 1; \frac{\hat{1}}{4}, -\frac{\hat{3}}{4}, \frac{\hat{3}}{4}, -\frac{\hat{1}}{4})$
		$S(4, 3, 2, 1) =$	$(-\frac{\hat{1}}{2}, 0, 1; -\frac{\hat{1}}{4}, \frac{\hat{3}}{4}, -\frac{\hat{3}}{4}, \frac{\hat{1}}{4})$
		$S(4, 3, 2, 2) =$	$(\frac{\hat{1}}{2}, 1, 0; \frac{\hat{3}}{4}, -\frac{\hat{1}}{4}, -\frac{\hat{3}}{4}, \frac{\hat{1}}{4})$
		$S(4, 3, 2, 3) =$	$(\frac{\hat{1}}{2}, 1, 0; -\frac{\hat{1}}{4}, \frac{\hat{3}}{4}, \frac{\hat{1}}{4}, -\frac{\hat{3}}{4})$
		$S(4, 3, 2, 4) =$	$(-\frac{\hat{1}}{2}, 0, 1; \frac{\hat{3}}{4}, -\frac{\hat{1}}{4}, \frac{\hat{1}}{4}, -\frac{\hat{3}}{4})$
$2 \cdot 2$	4	$S(4, 2, 2, 1) =$	$(\hat{0}, 1, 1; -\frac{\hat{1}}{2}, -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}, \frac{\hat{1}}{2})$
		$S(4, 2, 2, 2) =$	$(\hat{1}, 0, 0; \frac{\hat{1}}{2}, \frac{\hat{1}}{2}, \frac{\hat{1}}{2}, \frac{\hat{1}}{2})$
		$S(4, 2, 2, 3) =$	$(\hat{1}, 0, 0; -\frac{\hat{1}}{2}, -\frac{\hat{1}}{2}, -\frac{\hat{1}}{2}, -\frac{\hat{1}}{2})$
		$S(4, 2, 2, 4) =$	$(\hat{0}, 1, 1; \frac{\hat{1}}{2}, \frac{\hat{1}}{2}, -\frac{\hat{1}}{2}, -\frac{\hat{1}}{2})$
$1 \cdot 1 \cdot 1 \cdot 1$	2	$S(4, 4, 2, 1) =$	$(\hat{1}, 1, 1; \hat{0}, \hat{0}, \hat{0}, \hat{0})$
		$S(4, 4, 2, 2) =$	$(\hat{0}, 0, 0; \hat{1}, \hat{1}, \hat{0}, \hat{0})$
		$S(4, 4, 2, 3) =$	$(\hat{0}, 0, 0; \hat{0}, \hat{0}, \hat{1}, \hat{1})$
		$S(4, 4, 2, 4) =$	$(\hat{1}, 1, 1; \hat{1}, \hat{1}, \hat{1}, \hat{1})$

Cycles of Lengths n_i	Order N	Designation	Boundary Vector Components
5	10	→	$S(5, 1, 1, 1) = (\frac{1}{5}, \frac{3}{5}, 1; -\frac{4}{5}, -\frac{2}{5}, 0, 0, \frac{2}{5}, \frac{4}{5})$ $S(5, 1, 1, 2) = (-\frac{4}{5}, -\frac{2}{5}, 0; \frac{1}{5}, \frac{3}{5}, 1, 0, \frac{2}{5}, \frac{4}{5})$ $S(5, 1, 1, 3) = (-\frac{4}{5}, -\frac{2}{5}, 0; -\frac{4}{5}, -\frac{2}{5}, 0, 1, -\frac{3}{5}, -\frac{1}{5})$ $S(5, 1, 1, 4) = (\frac{1}{5}, \frac{3}{5}, 1; \frac{1}{5}, \frac{3}{5}, 1, 1, -\frac{3}{5}, \frac{1}{5})$ $S(5, 2, 1, 1) = (-\frac{3}{5}, \frac{1}{5}, 1; -\frac{3}{5}, \frac{1}{5}, 1, 1, -\frac{1}{5}, \frac{3}{5})$ $S(5, 2, 1, 2) = (\frac{2}{5}, -\frac{4}{5}, 0; \frac{2}{5}, -\frac{4}{5}, 0, 1, -\frac{1}{5}, \frac{3}{5})$ $S(5, 2, 1, 3) = (\frac{2}{5}, -\frac{4}{5}, 0; -\frac{3}{5}, \frac{1}{5}, 1, 0, \frac{4}{5}, -\frac{2}{5})$ $S(5, 2, 1, 4) = (-\frac{3}{5}, \frac{1}{5}, 1; \frac{2}{5}, -\frac{4}{5}, 0, 0, \frac{4}{5}, -\frac{2}{5})$ $S(5, 3, 1, 1) = (\frac{3}{5}, -\frac{1}{5}, 1; -\frac{2}{5}, \frac{4}{5}, 0, 0, -\frac{4}{5}, \frac{2}{5})$ $S(5, 3, 1, 2) = (-\frac{2}{5}, \frac{4}{5}, 0; \frac{3}{5}, -\frac{1}{5}, 1, 0, -\frac{4}{5}, \frac{2}{5})$ $S(5, 3, 1, 3) = (-\frac{2}{5}, \frac{4}{5}, 0; -\frac{2}{5}, \frac{4}{5}, 0, 1, \frac{1}{5}, -\frac{3}{5})$ $S(5, 3, 1, 4) = (\frac{3}{5}, -\frac{1}{5}, 1; \frac{3}{5}, -\frac{1}{5}, 1, 1, \frac{1}{5}, -\frac{3}{5})$ $S(5, 4, 1, 1) = (-\frac{1}{5}, -\frac{3}{5}, 1; -\frac{1}{5}, -\frac{3}{5}, 1, 1, \frac{3}{5}, \frac{1}{5})$ $S(5, 4, 1, 2) = (\frac{4}{5}, \frac{2}{5}, 0; \frac{4}{5}, \frac{2}{5}, 0, 1, \frac{3}{5}, \frac{1}{5})$ $S(5, 4, 1, 3) = (\frac{4}{5}, \frac{2}{5}, 0; -\frac{1}{5}, -\frac{3}{5}, 1, 0, -\frac{2}{5}, -\frac{4}{5})$ $S(5, 4, 1, 4) = (-\frac{1}{5}, -\frac{3}{5}, 1; \frac{4}{5}, \frac{2}{5}, 0, 0, -\frac{2}{5}, -\frac{4}{5})$
1 · 1 · 1 · 1 · 1	2	→	$S(5, 5, 1, 1) = (\hat{1}, \hat{1}, 1; \hat{0}, \hat{0}, 0, 0, \hat{0}, \hat{0})$ $S(5, 5, 1, 2) = (\hat{0}, \hat{0}, 0; \hat{1}, \hat{1}, 1, 0, \hat{0}, \hat{0})$ $S(5, 5, 1, 3) = (\hat{0}, \hat{0}, 0; \hat{0}, \hat{0}, 0, 1, \hat{1}, \hat{1})$ $S(5, 5, 1, 4) = (\hat{0}, \hat{0}, 0; \hat{0}, \hat{0}, 0, 1, \hat{1}, \hat{1})$

GSO projection classes for $S(5, 1, 1, 1)$ contribution to gravitino sector \mathbf{S}_{10}

- class (5.5.1.1): \mathbf{F} coef = 1 and $\frac{1}{2}\text{dp} = -\frac{3}{4} \pmod{\frac{1}{5}}$
 $\{S(5, 1, 1, 1), S(5, 1, 1, 4), S(5, 2, 1, 1), S(5, 2, 1, 4), S(5, 3, 1, 1), S(5, 3, 1, 4),$
 $S(5, 4, 1, 1), S(5, 4, 1, 4), S(5, 5, 1, 1), S(5, 5, 1, 4)\}$
class (5.5.1.2): \mathbf{F} coef = 0 and $\frac{1}{2}\text{dp} = 0 \pmod{\frac{1}{5}}$

$$\{S(5, 1, 1, 2), S(5, 1, 1, 3), S(5, 2, 1, 2), S(5, 2, 1, 3), S(5, 3, 1, 2), S(5, 3, 1, 3), \\ S(5, 4, 1, 2), S(5, 4, 1, 3), S(5, 5, 1, 2), S(5, 5, 1, 3)\}$$

GSO projection classes for $S(5, 3, 1, 1)$ contribution to gravitino sector $3\mathbf{S}_{10}$

Same \mathbf{F} coef and $\frac{1}{2}\text{dp}$ Classes as $3\mathbf{S}_{10}$ above.

GSO projection classes for $S(5, 5, 1, 1)$ contribution to gravitino sector \mathbf{S}_1

class (1.5.1.1):	\mathbf{F} coef =	$\hat{1}, \hat{1}, 1$	and $\frac{1}{2}\text{dp} =$	$-1\frac{1}{4}$	(mod 1)
		$\{S(5, 5, 1, 1), S(5, 5, 1, 4)\}$			
class (1.5.1.2):	\mathbf{F} coef =	$\hat{0}, \hat{0}, 0$	and $\frac{1}{2}\text{dp} =$	0	(mod 1)
		$\{S(5, 5, 1, 2), S(5, 5, 1, 3)\}$			
class (1.5.1.3):	\mathbf{F} coef =	$-\frac{3}{5}, -\frac{1}{5}, 1$	and $\frac{1}{2}\text{dp} =$	$\frac{9}{20}$	(mod 1)
		$\{S(5, 3, 1, 1), S(5, 3, 1, 4)\}$			
class (1.5.1.4):	\mathbf{F} coef =	$-\frac{3}{5}, \frac{1}{5}, 1$	and $\frac{1}{2}\text{dp} =$	$\frac{1}{20}$	(mod 1)
		$\{S(5, 2, 1, 1), S(5, 2, 1, 4)\}$			
class (1.5.1.5):	\mathbf{F} coef =	$\frac{1}{5}, \frac{3}{5}, 1$	and $\frac{1}{2}\text{dp} =$	$\frac{13}{20}$	(mod 1)
		$\{S(5, 1, 1, 1), S(5, 1, 1, 4)\}$			
class (1.5.1.6):	\mathbf{F} coef =	$-\frac{1}{5}, -\frac{3}{5}, 1$	and $\frac{1}{2}\text{dp} =$	$-\frac{3}{20}$	(mod 1)
		$\{S(5, 4, 1, 1), S(5, 4, 1, 4)\}$			
class (1.5.1.7):	\mathbf{F} coef =	$\frac{4}{5}, \frac{2}{5}, 0$	and $\frac{1}{2}\text{dp} =$	$\frac{3}{5}$	(mod 1)
		$\{S(5, 4, 1, 2), S(5, 4, 1, 3)\}$			
class (1.5.1.8):	\mathbf{F} coef =	$-\frac{4}{5}, -\frac{2}{5}, 0$	and $\frac{1}{2}\text{dp} =$	$-\frac{3}{5}$	(mod 1)
		$\{S(5, 1, 1, 2), S(5, 1, 1, 3)\}$			
class (1.5.1.9):	\mathbf{F} coef =	$\frac{2}{5}, -\frac{4}{5}, 0$	and $\frac{1}{2}\text{dp} =$	$-\frac{1}{5}$	(mod 1)
		$\{S(5, 2, 1, 2), S(5, 2, 1, 3)\}$			
class (1.5.1.10):	\mathbf{F} coef =	$-\frac{2}{5}, \frac{4}{5}, 0$	and $\frac{1}{2}\text{dp} =$	$\frac{1}{5}$	(mod 1)
		$\{S(5, 3, 1, 2), S(5, 3, 1, 3)\}$			

Cycles of Lengths n_i	Order N	Designation	Boundary Vector Components
5	10	$S(5, 1, 2, 1) =$	$(\frac{\hat{1}}{5}, \frac{\hat{3}}{5}, 1; -\frac{\hat{4}}{5}, \frac{\hat{2}}{5}, 0, 0, -\frac{\hat{2}}{5}, \frac{\hat{4}}{5})$
		$S(5, 1, 2, 2) =$	$(-\frac{\hat{4}}{5}, -\frac{\hat{2}}{5}, 0; \frac{\hat{1}}{5}, -\frac{\hat{3}}{5}, 1, 0, -\frac{\hat{2}}{5}, \frac{\hat{4}}{5})$
		$S(5, 1, 2, 3) =$	$(-\frac{\hat{4}}{5}, -\frac{\hat{2}}{5}, 0; -\frac{\hat{4}}{5}, \frac{\hat{2}}{5}, 0, 1, \frac{\hat{3}}{5}, -\frac{\hat{1}}{5})$
		$S(5, 1, 2, 4) =$	$(\frac{\hat{1}}{5}, \frac{\hat{3}}{5}, 1; \frac{\hat{1}}{5}, -\frac{\hat{3}}{5}, 1, 1, \frac{\hat{3}}{5}, \frac{\hat{1}}{5})$
		$S(5, 2, 2, 1) =$	$(-\frac{\hat{3}}{5}, \frac{\hat{1}}{5}, 1; -\frac{\hat{3}}{5}, -\frac{\hat{1}}{5}, 1, 1, \frac{\hat{1}}{5}, \frac{\hat{3}}{5})$
		$S(5, 2, 2, 2) =$	$(\frac{\hat{2}}{5}, -\frac{\hat{4}}{5}, 0; \frac{\hat{2}}{5}, \frac{\hat{4}}{5}, 0, 1, \frac{\hat{1}}{5}, \frac{\hat{3}}{5})$
		$S(5, 2, 2, 3) =$	$(\frac{\hat{2}}{5}, -\frac{\hat{4}}{5}, 0; -\frac{\hat{3}}{5}, -\frac{\hat{1}}{5}, 1, 0, -\frac{\hat{4}}{5}, -\frac{\hat{2}}{5})$
		$S(5, 2, 2, 4) =$	$(-\frac{\hat{3}}{5}, \frac{\hat{1}}{5}, 1; \frac{\hat{2}}{5}, \frac{\hat{4}}{5}, 0, 0, -\frac{\hat{4}}{5}, -\frac{\hat{2}}{5})$
		$S(5, 3, 2, 1) =$	$(\frac{\hat{3}}{5}, -\frac{\hat{1}}{5}, 1; -\frac{\hat{2}}{5}, -\frac{\hat{4}}{5}, 0, 0, \frac{\hat{4}}{5}, \frac{\hat{2}}{5})$
		$S(5, 3, 2, 2) =$	$(-\frac{\hat{2}}{5}, \frac{\hat{4}}{5}, 0; \frac{\hat{3}}{5}, \frac{\hat{1}}{5}, 1, 0, \frac{\hat{4}}{5}, \frac{\hat{2}}{5})$
		$S(5, 3, 2, 3) =$	$(-\frac{\hat{2}}{5}, \frac{\hat{4}}{5}, 0; -\frac{\hat{2}}{5}, -\frac{\hat{4}}{5}, 0, 0, -\frac{\hat{1}}{5}, -\frac{\hat{3}}{5})$
		$S(5, 3, 2, 4) =$	$(\frac{\hat{3}}{5}, -\frac{\hat{1}}{5}, 1; \frac{\hat{3}}{5}, \frac{\hat{1}}{5}, 1, 1, -\frac{\hat{1}}{5}, -\frac{\hat{3}}{5})$
		$S(5, 4, 2, 1) =$	$(-\frac{\hat{1}}{5}, -\frac{\hat{3}}{5}, 1; -\frac{\hat{1}}{5}, \frac{\hat{3}}{5}, 1, 1, -\frac{\hat{3}}{5}, \frac{\hat{1}}{5})$
		$S(5, 4, 2, 2) =$	$(\frac{\hat{4}}{5}, \frac{\hat{2}}{5}, 0; \frac{\hat{4}}{5}, -\frac{\hat{2}}{5}, 0, 1, -\frac{\hat{3}}{5}, \frac{\hat{1}}{5})$
		$S(5, 4, 2, 3) =$	$(\frac{\hat{4}}{5}, \frac{\hat{2}}{5}, 0; -\frac{\hat{1}}{5}, \frac{\hat{3}}{5}, 1, 0, \frac{\hat{2}}{5}, -\frac{\hat{4}}{5})$
		$S(5, 4, 2, 4) =$	$(-\frac{\hat{1}}{5}, -\frac{\hat{3}}{5}, 1; \frac{\hat{4}}{5}, -\frac{\hat{2}}{5}, 0, 0, \frac{\hat{2}}{5}, -\frac{\hat{4}}{5})$
$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$	2	$S(5, 5, 2, 1) =$	$(\hat{1}, \hat{1}, 1; \hat{0}, \hat{0}, 0, 0, \hat{0}, \hat{0})$
		$S(5, 5, 2, 2) =$	$(\hat{0}, \hat{0}, 0; \hat{1}, \hat{1}, 1, 0, \hat{0}, \hat{0})$
		$S(5, 5, 2, 3) =$	$(\hat{0}, \hat{0}, 0; \hat{0}, \hat{0}, 0, 1, \hat{1}, \hat{1})$
		$S(5, 5, 2, 4) =$	$(\hat{0}, \hat{0}, 0; \hat{0}, \hat{0}, 0, 1, \hat{1}, \hat{1})$

Cycles of Lengths n_i	Order N	Designation	Boundary Vector Components
6	12	$S(6, 1, 1, 1) = (\frac{1}{3}, \frac{2}{3}, 0, 1; -\frac{5}{6}, -\frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6})$ $S(6, 1, 1, 2) = (-\frac{2}{3}, -\frac{1}{3}, 1, 0; \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6})$ $S(6, 1, 1, 3) = (-\frac{2}{3}, -\frac{1}{3}, 1, 0; -\frac{5}{6}, -\frac{1}{2}, -\frac{1}{6}, -\frac{5}{6}, -\frac{1}{2}, -\frac{1}{6})$ $S(6, 1, 1, 4) = (\frac{1}{3}, \frac{2}{3}, 0, 1; \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, -\frac{5}{6}, -\frac{1}{2}, -\frac{1}{6})$ $S(6, 5, 1, 1) = (-\frac{1}{3}, -\frac{2}{3}, 0, 1; -\frac{1}{6}, -\frac{1}{2}, -\frac{5}{6}, \frac{5}{6}, \frac{1}{2}, \frac{1}{6})$ $S(6, 5, 1, 2) = (\frac{2}{3}, \frac{1}{3}, 1, 0; \frac{5}{6}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{2}, \frac{1}{6})$ $S(6, 5, 1, 3) = (\frac{2}{3}, \frac{1}{3}, 1, 0; -\frac{1}{6}, -\frac{1}{2}, -\frac{5}{6}, -\frac{1}{6}, -\frac{1}{2}, -\frac{5}{6})$ $S(6, 5, 1, 4) = (-\frac{1}{3}, -\frac{2}{3}, 0, 1; \frac{5}{6}, \frac{1}{2}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{2}, -\frac{5}{6})$	
3 · 3	6	$S(6, 2, 1, 1) = (-\frac{1}{3}, \frac{1}{3}, 1, 1; -\frac{2}{3}, \hat{0}, \frac{2}{3}, -\frac{2}{3}, \hat{0}, \frac{2}{3})$ $\rightarrow S(6, 2, 1, 2) = (\frac{2}{3}, -\frac{2}{3}, 0, 0; \frac{1}{3}, \hat{1}, -\frac{1}{3}, -\frac{2}{3}, \hat{0}, \frac{2}{3})$ $S(6, 2, 1, 3) = (\frac{2}{3}, -\frac{2}{3}, 0, 0; -\frac{2}{3}, \hat{0}, \frac{2}{3}, \frac{1}{3}, \hat{1}, -\frac{1}{3})$ $S(6, 2, 1, 4) = (-\frac{1}{3}, \frac{1}{3}, 1, 1; \frac{1}{3}, \hat{1}, -\frac{1}{3}, \frac{1}{3}, \hat{1}, -\frac{1}{3})$ $S(6, 4, 1, 1) = (\frac{1}{3}, -\frac{1}{3}, 1, 1; -\frac{1}{3}, \hat{1}, \frac{1}{3}, -\frac{1}{3}, \hat{1}, \frac{1}{3})$ $S(6, 4, 1, 2) = (-\frac{2}{3}, \frac{2}{3}, 0, 0; \frac{2}{3}, \hat{0}, -\frac{2}{3}, -\frac{1}{3}, \hat{1}, \frac{1}{3})$ $S(6, 4, 1, 3) = (-\frac{2}{3}, \frac{2}{3}, 0, 0; -\frac{1}{3}, \hat{1}, \frac{1}{3}, \frac{2}{3}, \hat{0}, -\frac{2}{3})$ $S(6, 4, 1, 4) = (\frac{1}{3}, -\frac{1}{3}, 1, 1; \frac{2}{3}, \hat{0}, -\frac{2}{3}, \frac{2}{3}, \hat{0}, -\frac{2}{3})$	
2 · 2 · 2	4	$S(6, 3, 1, 1) = (\hat{1}, \hat{0}, 0, 1; -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ $S(6, 3, 1, 2) = (\hat{0}, \hat{1}, 1, 0; \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ $S(6, 3, 1, 3) = (\hat{0}, \hat{1}, 1, 0; -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ $S(6, 3, 1, 4) = (\hat{1}, \hat{0}, 0, 1; \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	
1 · 1 · 1 · 1 · 1 · 1	2	$S(6, 6, 1, 1) = (\hat{1}, \hat{1}, 1, 1; \hat{0}, \hat{0}, \hat{0}, \hat{0}, \hat{0}, \hat{0})$ $\rightarrow S(6, 6, 1, 2) = (\hat{0}, \hat{0}, 0, 0; \hat{1}, \hat{1}, \hat{1}, \hat{0}, \hat{0}, \hat{0})$ $S(6, 6, 1, 3) = (\hat{0}, \hat{0}, 0, 0; \hat{0}, \hat{0}, \hat{0}, \hat{1}, \hat{1}, \hat{1})$ $S(6, 6, 1, 4) = (\hat{1}, \hat{1}, 1, 1; \hat{1}, \hat{1}, \hat{1}, \hat{1}, \hat{1}, \hat{1})$	

GSO projection classes for $S(6, 2, 1, 2)$ contribution to gravitino sector \mathbf{S}_7

class (3.6.2.1): \mathbf{F} coef = $\hat{1}$ and $\frac{1}{2}\text{dp} = -\frac{1}{2} \pmod{\frac{1}{3}}$
 $\{S(6, 2, 1, 2), S(6, 2, 1, 4), S(6, 4, 1, 1), S(6, 4, 1, 3), S(6, 6, 1, 2), S(6, 6, 1, 4)\}$

class (3.6.2.2): \mathbf{F} coef = $\hat{0}$ and $\frac{1}{2}\text{dp} = 0 \pmod{\frac{1}{3}}$
 $\{S(6, 2, 1, 1), S(6, 2, 1, 3), S(6, 4, 1, 2), S(6, 4, 1, 4), S(6, 6, 1, 1), S(6, 6, 1, 3)\}$

class (3.6.2.3): \mathbf{F} coef = $\frac{\hat{1}}{2}$ and $\frac{1}{2}\text{dp} = \frac{1}{4} \pmod{\frac{1}{3}}$
 $\{S(6, 1, 1, 1), S(6, 1, 1, 3), S(6, 3, 1, 2), S(6, 3, 1, 4), S(6, 5, 1, 1), S(6, 5, 1, 3)\}$

class (3.6.2.4): \mathbf{F} coef = $-\frac{\hat{1}}{2}$ and $\frac{1}{2}\text{dp} = -\frac{1}{4} \pmod{\frac{1}{3}}$
 $\{S(6, 1, 1, 2), S(6, 1, 1, 4), S(6, 3, 1, 1), S(6, 3, 1, 3), S(6, 5, 1, 2), S(6, 6, 1, 4)\}$

GSO projection classes for $S(6, 6, 1, 2)$ contribution to gravitino sector \mathbf{S}_1

class (1.6.2.1): \mathbf{F} coef = $\hat{1}, \hat{1}, \hat{1}$ and $\frac{1}{2}\text{dp} = -\frac{1}{2} \pmod{1}$
 $\{S(6, 6, 1, 2), S(6, 6, 1, 4)\}$

class (1.6.2.2): \mathbf{F} coef = $\hat{0}, \hat{0}, \hat{0}$ and $\frac{1}{2}\text{dp} = 0 \pmod{1}$
 $\{S(6, 6, 1, 1), S(6, 5, 1, 3)\}$

class (1.6.2.3): \mathbf{F} coef = $\frac{\hat{1}}{3}, \hat{1}, -\frac{\hat{1}}{3}$ and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{1}$
 $\{S(6, 2, 1, 2), S(6, 2, 1, 4)\}$

class (1.6.2.4): \mathbf{F} coef = $-\frac{\hat{1}}{3}, \hat{1}, \frac{\hat{1}}{3}$ and $\frac{1}{2}\text{dp} = \frac{1}{2} \pmod{1}$
 $\{S(6, 4, 1, 1), S(6, 4, 1, 4)\}$

class (1.6.2.5): \mathbf{F} coef = $\frac{\hat{2}}{3}, \hat{0}, -\frac{\hat{2}}{3}$ and $\frac{1}{2}\text{dp} = 0 \pmod{1}$
 $\{S(6, 4, 1, 2), S(6, 4, 1, 4)\}$

class (1.6.2.6): \mathbf{F} coef = $-\frac{\hat{2}}{3}, \hat{0}, \frac{\hat{2}}{3}$ and $\frac{1}{2}\text{dp} = 0 \pmod{1}$
 $\{S(6, 2, 1, 1), S(6, 2, 1, 3)\}$

class (1.6.2.7): \mathbf{F} coef = $\frac{\hat{5}}{6}, \frac{\hat{1}}{2}, \frac{\hat{1}}{6}$ and $\frac{1}{2}\text{dp} = \frac{3}{4} \pmod{1}$
 $\{S(6, 5, 1, 2), S(6, 5, 1, 4)\}$

class (1.6.2.8): \mathbf{F} coef = $-\frac{\hat{5}}{6}, -\frac{\hat{1}}{2}, -\frac{\hat{1}}{6}$ and $\frac{1}{2}\text{dp} = -\frac{3}{4} \pmod{1}$
 $\{S(6, 1, 1, 1), S(6, 1, 1, 3)\}$

class (1.6.2.9): \mathbf{F} coef = $\frac{\hat{1}}{2}, -\frac{\hat{1}}{2}, \frac{\hat{1}}{2}$ and $\frac{1}{2}\text{dp} = \frac{1}{4} \pmod{1}$
 $\{S(6, 3, 1, 2), S(6, 3, 1, 4)\}$

class (1.6.2.10): \mathbf{F} coef = $-\frac{\hat{1}}{2}, \frac{\hat{1}}{2}, -\frac{\hat{1}}{2}$ and $\frac{1}{2}\text{dp} = -\frac{1}{4} \pmod{1}$
 $\{S(6, 3, 1, 1), S(6, 3, 1, 4)\}$

class (1.6.2.11): \mathbf{F} coef = $\frac{\hat{1}}{6}, \frac{\hat{1}}{2}, \frac{\hat{5}}{6}$ and $\frac{1}{2}\text{dp} = \frac{3}{4} \pmod{1}$

$$\{S(6, 1, 1, 2), S(6, 1, 1, 4)\}$$

$$\text{class (1.6.2.12): } \mathbf{F} \text{ coef} = -\frac{\hat{1}}{6}, -\frac{\hat{1}}{2}, -\frac{\hat{5}}{6} \quad \text{and } \frac{1}{2}\text{dp} = -\frac{3}{4} \pmod{1}$$

$$\{S(6, 5, 1, 1), S(6, 5, 1, 3)\}$$

Cycles of Lengths n_i	Order N	Designation	Boundary Vector Components
6	12	$S(6, 1, 2, 1) =$	$(\frac{1}{3}, \frac{2}{3}, 0, 1; -\frac{5}{6}, \frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{2}, \frac{5}{6})$
		$S(6, 1, 2, 2) =$	$(-\frac{2}{3}, -\frac{1}{3}, 1, 0; \frac{1}{6}, -\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, -\frac{1}{2}, \frac{5}{6})$
		$S(6, 1, 2, 3) =$	$(-\frac{2}{3}, -\frac{1}{3}, 1, 0; -\frac{5}{6}, \frac{1}{2}, -\frac{1}{6}, -\frac{5}{6}, \frac{1}{2}, -\frac{1}{6})$
		$S(6, 1, 2, 4) =$	$(\frac{1}{3}, \frac{2}{3}, 0, 1; \frac{1}{6}, -\frac{1}{2}, \frac{5}{6}, -\frac{5}{6}, \frac{1}{2}, -\frac{1}{6})$
		$S(6, 5, 2, 1) =$	$(-\frac{1}{3}, -\frac{2}{3}, 0, 1; -\frac{1}{6}, \frac{1}{2}, -\frac{5}{6}, \frac{5}{6}, -\frac{1}{2}, \frac{1}{6})$
		$S(6, 5, 2, 2) =$	$(\frac{2}{3}, \frac{1}{3}, 1, 0; \frac{5}{6}, -\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, -\frac{1}{2}, \frac{1}{6})$
		$S(6, 5, 2, 3) =$	$(\frac{2}{3}, \frac{1}{3}, 1, 0; -\frac{1}{6}, \frac{1}{2}, -\frac{5}{6}, -\frac{1}{6}, \frac{1}{2}, -\frac{5}{6})$
		$S(6, 5, 2, 4) =$	$(-\frac{1}{3}, -\frac{2}{3}, 0, 1; \frac{5}{6}, -\frac{1}{2}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{2}, -\frac{5}{6})$
$3 \cdot 3$	6	$S(6, 2, 2, 1) =$	$(-\frac{1}{3}, \frac{1}{3}, 1, 1; -\frac{2}{3}, \hat{0}, \frac{2}{3}, -\frac{2}{3}, \hat{0}, \frac{2}{3})$
		$S(6, 2, 2, 2) =$	$(\frac{2}{3}, -\frac{2}{3}, 0, 0; \frac{1}{3}, \hat{1}, -\frac{1}{3}, -\frac{2}{3}, \hat{0}, \frac{2}{3})$
		$S(6, 2, 2, 3) =$	$(\frac{2}{3}, -\frac{2}{3}, 0, 0; -\frac{2}{3}, \hat{0}, \frac{2}{3}, \frac{1}{3}, \hat{1}, -\frac{1}{3})$
		$S(6, 2, 2, 4) =$	$(-\frac{1}{3}, \frac{1}{3}, 1, 1; \frac{1}{3}, \hat{1}, -\frac{1}{3}, \frac{1}{3}, \hat{1}, -\frac{1}{3})$
		$S(6, 4, 2, 1) =$	$(\frac{1}{3}, -\frac{1}{3}, 1, 1; -\frac{1}{3}, \hat{1}, \frac{1}{3}, -\frac{1}{3}, \hat{1}, \frac{1}{3})$
		$S(6, 4, 2, 2) =$	$(-\frac{2}{3}, \frac{2}{3}, 0, 0; \frac{2}{3}, \hat{0}, -\frac{2}{3}, -\frac{1}{3}, \hat{1}, \frac{1}{3})$
		$S(6, 4, 2, 3) =$	$(-\frac{2}{3}, \frac{2}{3}, 0, 0; -\frac{1}{3}, \hat{1}, \frac{1}{3}, \frac{2}{3}, \hat{0}, -\frac{2}{3})$
		$S(6, 4, 2, 4) =$	$(\frac{1}{3}, -\frac{1}{3}, 1, 1; \frac{2}{3}, \hat{0}, -\frac{2}{3}, \frac{2}{3}, \hat{0}, -\frac{2}{3})$
$2 \cdot 2 \cdot 2$	4	$S(6, 3, 2, 1) =$	$(\hat{1}, \hat{0}, 0, 1; -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
		$S(6, 3, 2, 2) =$	$(\hat{0}, \hat{1}, 1, 0; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
		$S(6, 3, 2, 3) =$	$(\hat{0}, \hat{1}, 1, 0; -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
		$S(6, 3, 2, 4) =$	$(\hat{1}, \hat{0}, 0, 1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$	2	$S(6, 6, 2, 1) =$	$(\hat{1}, \hat{1}, 1, 1; \hat{0}, \hat{0}, \hat{0}, \hat{0}, \hat{0}, \hat{0})$
		$S(6, 6, 2, 2) =$	$(\hat{0}, \hat{0}, 0, 0; \hat{1}, \hat{1}, \hat{1}, \hat{0}, \hat{0}, \hat{0})$
		$S(6, 6, 2, 3) =$	$(\hat{0}, \hat{0}, 0, 0; \hat{0}, \hat{0}, \hat{0}, \hat{1}, \hat{1}, \hat{1})$
		$S(6, 6, 2, 4) =$	$(\hat{1}, \hat{1}, 1, 1; \hat{1}, \hat{1}, \hat{1}, \hat{1}, \hat{1}, \hat{1})$

Cycles of Lengths n_i	Order N	Designation	Boundary Vector Components
6	12	$S(6, 1, 3, 1) =$	$(\frac{1}{3}, \frac{2}{3}, 0, 1; -\frac{5}{6}, -\frac{1}{2}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{2}, \frac{5}{6})$
		$S(6, 1, 3, 2) =$	$(-\frac{2}{3}, -\frac{1}{3}, 1, 0; \frac{1}{6}, \frac{1}{2}, -\frac{5}{6}, -\frac{1}{6}, \frac{1}{2}, \frac{5}{6})$
		$S(6, 1, 3, 3) =$	$(-\frac{2}{3}, -\frac{1}{3}, 1, 0; -\frac{5}{6}, -\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, -\frac{1}{2}, -\frac{1}{6})$
		$S(6, 1, 3, 4) =$	$(\frac{1}{3}, \frac{2}{3}, 0, 1; \frac{1}{6}, \frac{1}{2}, -\frac{5}{6}, \frac{5}{6}, -\frac{1}{2}, -\frac{1}{6})$
		$S(6, 5, 3, 1) =$	$(-\frac{1}{3}, -\frac{2}{3}, 0, 1; -\frac{1}{6}, -\frac{1}{2}, \frac{5}{6}, -\frac{5}{6}, \frac{1}{2}, \frac{1}{6})$
		$S(6, 5, 3, 2) =$	$(\frac{2}{3}, \frac{1}{3}, 1, 0; \frac{5}{6}, \frac{1}{2}, -\frac{1}{6}, -\frac{5}{6}, \frac{1}{2}, \frac{1}{6})$
		$S(6, 5, 3, 3) =$	$(\frac{2}{3}, \frac{1}{3}, 1, 0; -\frac{1}{6}, -\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, -\frac{1}{2}, -\frac{5}{6})$
		$S(6, 5, 3, 4) =$	$(-\frac{1}{3}, -\frac{2}{3}, 0, 1; \frac{5}{6}, \frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{2}, -\frac{5}{6})$
$3 \cdot 3$	6	$S(6, 2, 3, 1) =$	$(-\frac{1}{3}, \frac{1}{3}, 1, 1; -\frac{2}{3}, \hat{0}, -\frac{2}{3}, \frac{2}{3}, \hat{0}, \frac{2}{3})$
		$S(6, 2, 3, 2) =$	$(\frac{2}{3}, -\frac{2}{3}, 0, 0; \frac{1}{3}, \hat{1}, \frac{1}{3}, \frac{2}{3}, \hat{0}, \frac{2}{3})$
		$S(6, 2, 3, 3) =$	$(\frac{2}{3}, -\frac{2}{3}, 0, 0; -\frac{2}{3}, \hat{0}, -\frac{2}{3}, -\frac{1}{3}, \hat{1}, -\frac{1}{3})$
		$S(6, 2, 3, 4) =$	$(-\frac{1}{3}, \frac{1}{3}, 1, 1; \frac{1}{3}, \hat{1}, \frac{1}{3}, -\frac{1}{3}, \hat{1}, -\frac{1}{3})$
		$S(6, 4, 3, 1) =$	$(\frac{1}{3}, -\frac{1}{3}, 1, 1; -\frac{1}{3}, \hat{1}, -\frac{1}{3}, \frac{1}{3}, \hat{1}, \frac{1}{3})$
		$S(6, 4, 3, 2) =$	$(-\frac{2}{3}, \frac{2}{3}, 0, 0; \frac{2}{3}, \hat{0}, \frac{2}{3}, \frac{1}{3}, \hat{1}, \frac{1}{3})$
		$S(6, 4, 3, 3) =$	$(-\frac{2}{3}, \frac{2}{3}, 0, 0; -\frac{1}{3}, \hat{1}, -\frac{1}{3}, -\frac{2}{3}, \hat{0}, -\frac{2}{3})$
		$S(6, 4, 3, 4) =$	$(\frac{1}{3}, -\frac{1}{3}, 1, 1; \frac{2}{3}, \hat{0}, \frac{2}{3}, -\frac{2}{3}, \hat{0}, -\frac{2}{3})$
$2 \cdot 2 \cdot 2$	4	$S(6, 3, 3, 1) =$	$(\hat{1}, \hat{0}, 0, 1; -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$
		$S(6, 3, 3, 2) =$	$(\hat{0}, \hat{1}, 1, 0; \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$
		$S(6, 3, 3, 3) =$	$(\hat{0}, \hat{1}, 1, 0; -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$
		$S(6, 3, 3, 4) =$	$(\hat{1}, \hat{0}, 0, 1; \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$
$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$	2	$S(6, 6, 3, 1) =$	$(\hat{1}, \hat{1}, 1, 1; \hat{0}, \hat{0}, \hat{0}, \hat{0}, \hat{0}, \hat{0})$
		$S(6, 6, 3, 2) =$	$(\hat{0}, \hat{0}, 0, 0; \hat{1}, \hat{1}, \hat{1}, \hat{0}, \hat{0}, \hat{0})$
		$S(6, 6, 3, 3) =$	$(\hat{0}, \hat{0}, 0, 0; \hat{0}, \hat{0}, \hat{0}, \hat{1}, \hat{1}, \hat{1})$
		$S(6, 6, 3, 4) =$	$(\hat{1}, \hat{1}, 1, 1; \hat{1}, \hat{1}, \hat{1}, \hat{1}, \hat{1}, \hat{1})$

Cycles of Lengths n_i	Order N	Designation	Boundary Vector Components
6	12	$S(6, 1, 4, 1) =$	$(\frac{1}{3}, \frac{2}{3}, 0, 1; -\frac{5}{6}, +\frac{1}{2}, +\frac{1}{6}, -\frac{1}{6}, -\frac{1}{2}, \frac{5}{6})$
		$S(6, 1, 4, 2) =$	$(-\frac{2}{3}, -\frac{1}{3}, 1, 0; \frac{1}{6}, -\frac{1}{2}, -\frac{5}{6}, -\frac{1}{6}, -\frac{1}{2}, \frac{5}{6})$
		$S(6, 1, 4, 3) =$	$(-\frac{2}{3}, -\frac{1}{3}, 1, 0; -\frac{5}{6}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{2}, -\frac{1}{6})$
		$S(6, 1, 4, 4) =$	$(\frac{1}{3}, \frac{2}{3}, 0, 1; \frac{1}{6}, -\frac{1}{2}, -\frac{5}{6}, \frac{5}{6}, \frac{1}{2}, -\frac{1}{6})$
		$S(6, 5, 4, 1) =$	$(-\frac{1}{3}, -\frac{2}{3}, 0, 1; -\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, -\frac{5}{6}, \frac{1}{2}, \frac{1}{6})$
		$S(6, 5, 4, 2) =$	$(\frac{2}{3}, \frac{1}{3}, 1, 0; \frac{5}{6}, -\frac{1}{2}, -\frac{1}{6}, -\frac{5}{6}, -\frac{1}{2}, \frac{1}{6})$
		$S(6, 5, 4, 3) =$	$(\frac{2}{3}, \frac{1}{3}, 1, 0; -\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{2}, -\frac{5}{6})$
		$S(6, 5, 4, 4) =$	$(-\frac{1}{3}, -\frac{2}{3}, 0, 1; \frac{5}{6}, -\frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, -\frac{5}{6})$
$3 \cdot 3$	6	$S(6, 2, 4, 1) =$	$(-\frac{1}{3}, \frac{1}{3}, 1, 1; -\frac{2}{3}, \hat{0}, -\frac{2}{3}, \frac{2}{3}, \hat{0}, \frac{2}{3})$
		$S(6, 2, 4, 2) =$	$(\frac{2}{3}, -\frac{2}{3}, 0, 0; \frac{1}{3}, \hat{1}, \frac{1}{3}, \frac{2}{3}, \hat{0}, \frac{2}{3})$
		$S(6, 2, 4, 3) =$	$(\frac{2}{3}, -\frac{2}{3}, 0, 0; -\frac{2}{3}, \hat{0}, -\frac{2}{3}, -\frac{1}{3}, \hat{1}, -\frac{1}{3})$
		$S(6, 2, 4, 4) =$	$(-\frac{1}{3}, \frac{1}{3}, 1, 1; \frac{1}{3}, \hat{1}, \frac{1}{3}, -\frac{1}{3}, \hat{1}, -\frac{1}{3})$
		$S(6, 4, 4, 1) =$	$(\frac{1}{3}, -\frac{1}{3}, 1, 1; -\frac{1}{3}, \hat{1}, -\frac{1}{3}, \frac{1}{3}, \hat{1}, \frac{1}{3})$
		$S(6, 4, 4, 2) =$	$(-\frac{2}{3}, \frac{2}{3}, 0, 0; \frac{2}{3}, \hat{0}, \frac{2}{3}, \frac{1}{3}, \hat{1}, \frac{1}{3})$
		$S(6, 4, 4, 3) =$	$(-\frac{2}{3}, \frac{2}{3}, 0, 0; -\frac{1}{3}, \hat{1}, -\frac{1}{3}, -\frac{2}{3}, \hat{0}, -\frac{2}{3})$
		$S(6, 4, 4, 4) =$	$(\frac{1}{3}, -\frac{1}{3}, 1, 1; \frac{2}{3}, \hat{0}, \frac{2}{3}, -\frac{2}{3}, \hat{0}, -\frac{2}{3})$
$2 \cdot 2 \cdot 2$	4	$S(6, 3, 4, 1) =$	$(\hat{1}, \hat{0}, 0, 1; -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
		$S(6, 3, 4, 2) =$	$(\hat{0}, \hat{1}, 1, 0; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
		$S(6, 3, 4, 3) =$	$(\hat{0}, \hat{1}, 1, 0; -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
		$S(6, 3, 4, 4) =$	$(\hat{1}, \hat{0}, 0, 1; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$	2	$S(6, 6, 4, 1) =$	$(\hat{1}, \hat{1}, 1, 1; \hat{0}, \hat{0}, \hat{0}, \hat{0}, \hat{0}, \hat{0})$
		$S(6, 6, 4, 2) =$	$(\hat{0}, \hat{0}, 0, 0; \hat{1}, \hat{1}, \hat{1}, \hat{0}, \hat{0}, \hat{0})$
		$S(6, 6, 4, 3) =$	$(\hat{0}, \hat{0}, 0, 0; \hat{0}, \hat{0}, \hat{0}, \hat{1}, \hat{1}, \hat{1})$
		$S(6, 6, 4, 4) =$	$(\hat{1}, \hat{1}, 1, 1; \hat{1}, \hat{1}, \hat{1}, \hat{1}, \hat{1}, \hat{1})$

References

- [1] K. S. Narain, *Phys. Lett.* **B169** (1986) 41.
- [2] W. Lerche, D. Lüst, and A.N. Schellekens, *Nucl. Phys.* **B287** (1987) 477.
- [3] I. Antoniadis, C. Bachas, C. Kounnas, and P. Windey, *Phys. Lett.* **B171** (1986) 51;
I. Antoniadis, C. Bachas, and C. Kounnas, *Nucl. Phys.* **B289** (1987) 87;
I. Antoniadis and C. Bachas, *Nucl. Phys.* **B298** (1988) 586.
- [4] H. Kawai, D. Lewellen, and S.-H. Tye, *Nucl. Phys.* **B288** (1987) 1;
H. Kawai, D. Lewellen, J.A. Schwartz, and S.-H. Tye, *Nucl. Phys.* **B299** (1988) 431.
- [5] L. Dixon, J. Harvey, C. Vafa and E. Witten, *Nucl. Phys.* **B261** (1985) 651; **B274** (1986) 285.
- [6] L. Dixon, V. Kaplunovsky, and C. Vafa, *Nucl. Phys.* **B294** (1987) 443.
- [7] K. S. Narain, M.H. Sarmadi, and C. Vafa, *Nucl. Phys.* **B288** (1987) 551.
- [8] P. Candelas, G. Horowitz, A. Strominger, and E. Witten, *Nucl. Phys.* **B258** (1985) 46.
- [9] P. Candelas, A.M. Dale, C.A. Lutken, and R. Schimmrigk, *Nucl. Phys.* **B298** (1988) 493.
- [10] D. Gepner, *Nucl. Phys.* **B296** (1988) 757.
- [11] P. Langacker, *Phys. Rev.* **D47** (1993) 4028.
- [12] H. Dreiner, J. Lopez, D.V. Nanopoulos, and D. Reiss, *Nucl. Phys.* **B320** (1989) 401.
- [13] A. Faraggi, D.V. Nanopoulos, and K. Yuan, *Nucl. Phys.* **B335** (1990) 347.
A. Faraggi, *Nucl. Phys.* **B387** (1992) 239.
A. Faraggi and D.V. Nanopoulos, *Phys. Rev.* **D48** (1993) 3288.
I. Antoniadis, J. Ellis, J. Hagelin, and D.V. Nanopoulos, *Phys. Lett.* **B194** (1987) 231.
- [14] I. Antoniadis and C. Bachas, *Nucl. Phys.* **B298** (1988) 586.
- [15] D. Bailin, *Mod. Phys. Lett.* **4** (1989) 2339.
D. Bailin, D. Dunbar, and A. Love, *Nucl. Phys.* **B330** (1990) 124; *Int. Jour. Mod. Phys.* **A5** (1990) 939.
D. Bailin and A. Love, *Phys. Lett.* **B292** (1992) 315.
- [16] G. Cleaver, Unpublished research.
- [17] D. Lüst and S. Theisen, Lectures on String Theory, (Springer-Verlag, Berlin, 1989).
- [18] S. Kalara, J. Lopez, and D.V. Nanopoulos, *Phys. Lett.* **B245** (1990) 421.
- [19] I. Giannakis, D.V. Nanopoulos, and K. Yuan, *Phys. Rev.* **D52** (1995) 1026.
- [20] G. Cleaver, “*GUT’s With Adjoint Higgs Fields From Superstrings*,” OHSTPY-HEP-T-94-007. In the Proceedings of PASCOS ’94, Syracuse, New York, 1994.
- [21] S. Chaudhuri, S. Chung, and J. Lykken, “*Fermion Masses from Superstring Models with Adjoint Scalars*,” Fermilab-PUB-94/137-T; NSF-ITP-94-50.

- [22] G. Cleaver, “*Free Fermionic Solutions To $N=1$ SUSY and Three Generations*”; “*The Search For Stringy $SO(10)$ SUSY-GUTS.*” Research in Progress.